Graphs

A graph consists of a collection of objects called vertices plus a collection of pairs of vertices called edges.

A graph is usually pictured by representing each vertex by a point and each edge (pair of vertices) by a line connecting the corresponding points.

e.g. The 'map of Australia' has vertices \{ T, V, SA, NSW, WA, NT, Q, ACT \} and edges \{(SA, WA), (SA, NT), (SA, Q), (SA, NSW), (SA, V), (WA, NT), (NT, Q), (Q, NSW), (V, NSW), (ACT, NSW) \}.

Some language for graphs

We allow the possibility that there is more than one edge between two vertices. We then say that the graph has multiple edges.

We allow the possibility that an edge connects a vertex to itself. We call such an edge a loop.

We sometimes allow that each edge in a graph can be given a direction—from one vertex to another—we then say that we have a directed graph.
Walking through graphs

A path in an (undirected) graph is a sequence of edges of the form

\((a_1, a_2), (a_2, a_3), \ldots, (a_{n-1}, a_n), (a_n, a_{n+1})\).

Thus the second vertex of each edge is the first vertex of the next.

We can also describe the path by listing its vertices in order.

The length of the path is the number of edges it involves.

We allow the possibility that each edge of the graph is assigned a number. We then say that we have a weighted graph.

If each edge of the graph is assigned a label—for example, some letter of the alphabet—we say that we have a labelled graph.
A closed path is one where the first and last vertices coincide. It is sometimes called a circuit.

A cycle is a closed path with vertices $a_1, a_2, \ldots, a_n, a_{n+1} = a_1$ where $a_i \neq a_j$ if $i \neq j$ (except for $a_{n+1} = a_1$). That is, we visit no vertex on the path more than once, except for the starting vertex. It should also not repeat any edges.

A reduced path is one with no part of the form $a \rightarrow b \rightarrow a$ i.e. no 'doubling back'.

An Euler path is one which uses each edge of the graph exactly once.

An Euler circuit is an Euler path which is also a circuit.
A graph is connected if for any two vertices \(a\) and \(b\) there is a path starting at \(a\) and ending at \(b\).

A connected graph which has no cycles is called a tree.

Note that a graph with no cycles can have neither loops nor multiple edges.

A tree can also be described as a graph in which between any two points there is a unique reduced path.

The degree of a vertex is the number of edges at that vertex. If there is a loop at the vertex, this adds 2 to the degree.

If an edge "meets" a vertex, we say the edge and the vertex are incident.