Exam duration: Three hours

Reading time: 15 minutes

This paper has 7 pages

Common Content: This examination paper contains questions in common with the paper for 620-122.

Authorized Materials: No materials are authorized. Calculators and mathematical tables are not permitted. Candidates are reminded that no written or printed material related to the subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators: One 14 page script book is to be given to each student initially. Students may retain this examination paper. No written or printed material related to the subject may be brought into the examination. No mathematical tables or calculators may be used.

Instructions to Students: This examination consists of 11 questions. All questions may all be attempted. The number of marks for each question is indicated on the examination paper. The total number of marks is 85. Use of calculators is neither allowed nor is it necessary for a successful completion of this examination paper.

Paper to be held by Baillieu Library: This paper may be reproduced and lodged with the Baillieu Library.
1. Let
\[
A = \begin{bmatrix}
1 & -1 & 2 & 2 \\
2 & 0 & 1 & 0 \\
5 & -3 & 7 & -6 \\
1 & 1 & -1 & 3
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 & 1/2 & 0 \\
0 & 1 & -3/2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]
You are given that the matrix \( B \) is the reduced row echelon form of the matrix \( A \). Using this information, or otherwise, answer the following.

(a) What is the rank of \( A \)?
(b) Write down a basis for the column space of \( A \).
(c) Write down a basis for the row space of \( A \).
(d) Use your answer to (a) to compute the dimension of the solution space of \( A \).
(e) Find a basis for the solution space of \( A \).

[9 marks]

2. (a) Which of the following are subspaces of the vector space \( \mathbb{P}_2 \) of polynomials of degree \( \leq 2 \)? Explain your answers by either verifying the appropriate axioms, or providing a counter example.

(i) \( V = \{a + bx + cx^2 : a = c\} \)
(ii) \( U = \{a + bx + cx^2 : b^2 - 4ac = 0\} \)

[4 marks]

(b) Let \( S, T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be linear transformations of the plane where \( S \) is the rotation by 180° anticlockwise about the origin and \( T \) is the reflection in the line \( y = 0 \). Find the standard matrices for

(i) \( S \), (ii) \( T \), (iii) \( T \) followed by \( S \).

[4 marks]

Please turn over ...
3. (a) (i) Using $\mathbb{Z}_2$ arithmetic, find the solution space of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

(ii) Is 1011 a valid code word with respect to the check matrix $A$ in (i)?

(b) A small square paddock of area $20 \times 20$ square metres is given $xy$-coordinates so that its centre is at $(0,0)$, its bottom left corner at $(-10,-10)$, and its top right corner at $(10,10)$. It is desired to dig a narrow straight trench across the paddock so that it passes as close as possible to the points specified by the data

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-4$</th>
<th>$1$</th>
<th>$3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$-2$</td>
<td>$1$</td>
<td>$2$</td>
</tr>
</tbody>
</table>

Use the method of least squares to find the equation of the trench. What is the $y$ coordinate, to the nearest metre, of the trench at the right hand boundary of the paddock?

4. (a) (i) Write the formula

$$\langle x, y \rangle = \langle (x_1, x_2), (y_1, y_2) \rangle = 4x_1y_1 - 2x_1y_2 - x_2y_1 + 5x_2y_2$$

in the form

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

for certain $a, b, c, d$.

(ii) From your answer to (i), do you suspect the formula given in (i) defines an inner product on $\mathbb{R}^2$? Prove this by either verifying the 4 axioms, or providing a counter example.

(b) It is known that

$$\langle x, y \rangle = \langle (x_1, x_2), (y_1, y_2) \rangle = 4x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2$$

is an inner product on $\mathbb{R}^2$.

(i) Decide whether the vectors $(1, 0)$ and $(0, 1)$ are orthogonal relative to the above inner product.

(ii) Calculate the norm of $(3, 4)$ with respect to the above inner product.
5. (a) Use the Gram-Schmidt procedure to find an orthonormal basis for the subspace $W$ of $\mathbb{R}^5$ spanned by the vectors

$$(-1, -1, 1, 0, 0), \quad (0, -1, 0, 0, 1).$$

(Use the dot product on $\mathbb{R}^5$ as the inner product.)

[4 marks]

(b) Find the orthogonal projection of the vector $u = (-2, 4, 3, 0, 0)$ onto the subspace $W$ of (a).

[2 marks]

6. Consider the matrix

$$C = \begin{bmatrix} 0.98 & 0.01 \\ 0.02 & 0.99 \end{bmatrix}.$$

(a) Find an invertible matrix $P$ and a diagonal matrix $D$ such that $C = PDP^{-1}$. [Hint: To find the eigenvalues use an elementary row operation to simplify $C - \lambda I$.]

[4 marks]

(b) Use your results from (a) to find a formula for $C^n$ valid for each integer $n \geq 1$.

[2 marks]

(c) Describe the limiting behaviour of $C^n$ as $n \to \infty$.

[1 mark]

Please turn over . . .
7. You are given a linear transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) where
\[
T(x, y) = (2x + y, x + 2y),
\]
the basis \( \mathcal{B} = \{(1, 1), (-1, 1)\} \) for \( \mathbb{R}^2 \), and the standard basis \( \mathcal{S} = \{(1, 0), (0, 1)\} \) for \( \mathbb{R}^2 \).

(a) Show that \( T \) satisfies the two axioms required of a linear transformation.

(b) Write down the standard matrix representation of \( T \), \( [T]_S \).

(c) Calculate the image of \( T \).

(d) Find the change of basis matrix (transition matrix) \( P_{\mathcal{B}\to\mathcal{S}} \) from \( \mathcal{B} \) to \( \mathcal{S} \), and also find the change of basis matrix (transition matrix) \( P_{\mathcal{S}\to\mathcal{B}} \) from \( \mathcal{S} \) to \( \mathcal{B} \).

(e) Use your transition matrices to find the matrix representation \( [T]_\mathcal{B} \) for \( T \) with respect to the basis \( \mathcal{B} \).

(f) Use your result from part (e) to write down the eigenvalues and corresponding eigenvectors of \( [T]_S \).
8. A conic is given by the equation

\[ 5x^2 + 4xy + 5y^2 = 9. \]

(a) Rewrite the equation in matrix form \( \mathbf{x}^T A \mathbf{x} = 9 \), where \( A \) is a symmetric matrix and \( \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \).

[1 mark]

(b) Find the eigenvalues and eigenvectors of \( A \).

[3 marks]

(c) Write down the equation for the conic in standard form, and thus identify the conic.

[2 marks]

In the following questions you may use the following standard limits. Note that \( \log n \) denotes the natural logarithm of \( n \).

(i) \( \lim_{n \to \infty} \frac{1}{n^p} = 0 \) \((p > 0)\)

(ii) \( \lim_{n \to \infty} r^n = 0 \) \((|r| < 1)\)

(iii) \( \lim_{n \to \infty} a^{1/n} = 1 \) \((a > 0)\)

(iv) \( \lim_{n \to \infty} n^{1/n} = 1 \)

(v) \( \lim_{n \to \infty} \frac{a^n}{n!} = 0 \) \((a > 0)\)

(vi) \( \lim_{n \to \infty} \frac{\log n}{n^p} = 0 \) \((p > 0)\)

(vii) \( \lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^n = e^a \)

(viii) \( \lim_{n \to \infty} \frac{n^p}{a^n} = 0 \) \((p > 0, a > 1)\)

9. Determine which of the following sequences converge and which diverge. Find the limit for each convergent sequence. Please give complete arguments using the standard limits, arithmetic of limits, and other relevant theorems.

(a) \( a_n = \frac{n^3 + 8n}{2n^3 - 1} \)

(b) \( b_n = (3^n + 4^n)^{1/n} \)

(c) \( c_n = \log \left(\frac{n + 1}{n}\right) \)

(d) \( d_n = \frac{3^n}{(2^n + 1)} \)

[8 marks]

Please turn over ...
10. (a) For each of the following series explain either why the series diverges or find the $k$th partial sum of the series and use this to find the sum of the series.

$$\sum_{n=1}^{\infty} \frac{(-5)^n}{7^n} \quad \text{(i)} \quad \sum_{n=1}^{\infty} \frac{1}{n^{1/n}} \quad \text{(ii)}$$

(b) Find the radius of convergence of

$$\sum_{n=1}^{\infty} \frac{x^n}{2^n n^3}.$$

Also find the interval of convergence. Justify your conclusions by referring to appropriate tests.

11. (a) Find the Maclaurin series for $\sin x$.

(b) Use your answer to (a) to verify the formula

$$\frac{d^2}{dx^2} \sin x = -\sin x,$$

making sure you justify your working by referring to an appropriate theorem.

(c) Write down the third degree Taylor approximation to $\sin x$ about $x = 0$, and use this to estimate $\sin 0.3$. 

END OF EXAMINATION PAPER