Exam duration — Three hours
Reading time — 15 minutes
This paper consists of 3 pages

Identical Examination Papers: None.

Authorized Materials: No materials are authorized.

Mathematical tables and calculators are not permitted. Candidates are reminded that no written or printed material related to the subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators: Script books only are required. The students may remove the exam paper at the conclusion of the examination. No written or printed material related to the subject may be brought into the examination.

Instructions to Students: This examination consists of six questions. All questions may be answered.

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1. (a) If \( z_1 = -2 + 2i \) and \( z_2 = 1 + i\sqrt{3} \) show that for principal values
\[
\text{Arg}(z_1 z_2) \neq \text{Arg} z_1 + \text{Arg} z_2
\]

(b) State carefully the Fundamental Theorem of Algebra and apply it to determine the number of roots of the polynomial equation
\[
z^5 - z^4 - z^3 + z^2 + z - 1 = 0
\]
in \( \mathbb{C} \). Find these roots and hence factorize this polynomial into linear and quadratic factors with real coefficients.

1. \textbf{Solution}.

(a) In polar form
\[
z_1 = 2\sqrt{2}e^{3\pi i/4}, \quad z_2 = 2e^{\pi i/3}, \quad z_1 z_2 = 4\sqrt{2}e^{13\pi i/12} = 4\sqrt{2}e^{-11\pi i/12}
\]

Hence, since \(-\pi < \text{Arg} z \leq \pi\),
\[
\text{Arg} z_1 + \text{Arg} z_2 = \frac{3\pi}{4} + \frac{\pi}{3} = \frac{13\pi}{12} \neq -\frac{11\pi}{12} = \text{Arg}(z_1 z_2)
\]

(b) The Fundamental Theorem of Algebra states that a polynomial equation of degree \( n \geq 1 \) with complex coefficients has exactly \( n \) roots in \( \mathbb{C} \) counting multiplicities. Consequently this polynomial equation has 5 solutions in \( \mathbb{C} \). The integer factors of the constant coefficient \( a_0 = -1 \) are \( \pm 1 \). This gives \( z = \pm 1 \) as candidate roots of the polynomial \( P(z) \). We find that \( P(1) = 0 \) so that \( (z - 1) \) is a factor and so the polynomial factorizes
\[
P(z) = (z - 1)(z^4 - z^2 + 1) = 0
\]
Solving the quadratic in \( w = z^2 \) gives
\[
z^2 = \frac{1 \pm i\sqrt{3}}{2} = e^{\pi i/3} \quad \Rightarrow \quad z = \pm e^{\pm i\pi/6}
\]

Hence the five roots are \( z = 1, \pm e^{\pm i\pi/6} \) and the factorization into linear and quadratic factors with real coefficients is
\[
P(z) = (z - 1)(z - e^{\pi i/6})(z - e^{-\pi i/6})(z + e^{\pi i/6})(z + e^{-\pi i/6})
\]
\[
= (z - 1)(z^2 - \sqrt{3}z + 1)(z^2 + \sqrt{3}z + 1)
\]
2. Evaluate and fully simplify the following expressions if they exist, otherwise, explain why they do not exist. Carefully justify your arguments.

(a) \( \lim_{z \to \infty} z \sinh \frac{2}{z} \)

(b) \( \lim_{z \to i} (1 + i)^z \) (principal value)

(c) \( \arcsin(-i) \) (principal value)

(d) \( \lim_{n \to \infty} n i^n \)

2. Solution.

(a) By definition of the limit as \( z \to \infty \) and the strong form of l’Hôpital’s rule

\[
\lim_{z \to \infty} z \sinh \frac{2}{z} = \lim_{z \to 0} \frac{\sinh 2}{z} = \lim_{z \to 0} \frac{2 \cosh 2}{1} = 2
\]

(b) By definition of the complex power and continuity of the exponential

\[
\lim_{z \to i} (1 + i)^z = \lim_{z \to i} e^{z \log(1 + i)} = e^{i(\log \sqrt{2} + \pi i/4)} = e^{-\frac{\pi}{4} + \frac{1}{2} \log 2}
\]

(c) The definition is

\[
\arcsin z = -i \log[iz + e^{\frac{1}{2} \log(1 - z^2)}]
\]

So

\[
\arcsin(-i) = -i \log[1 + e^{\frac{1}{2} \log 2}] = -i \log(1 + \sqrt{2})
\]

(d) Taking the limit in subsequences with \( n = 4k \) or \( n = 4k + 2 \) gives two different results so the limit does not exist

\[
\lim_{k \to \infty} 4k(i)^{4k} = 4 \lim_{k \to \infty} k = \infty, \quad \lim_{k \to \infty} (4k+2)(i)^{4k+2} = -\lim_{k \to \infty} (4k+2) = -\infty
\]
3. (a) Clearly state Green’s theorem in the plane and use it to show that
\[ A = \frac{1}{2i} \oint_C z \, dz \]
is the area enclosed by the simple closed contour \( C \).

(b) Use this result to find the area enclosed between the parabola \( y = 1 - x^2 \) and the \( x \)-axis by evaluating a suitable closed contour integral in the complex plane.

(c) State clearly Cauchy’s theorem and why it does or does not apply to this contour integral.

3. Solution.

(a) Green’s theorem in the plane \( \mathbb{R}^2 \) states that if \( A \) is a closed region of the \( xy \)-plane bounded by a simple closed curve \( C \) and if \( M(x, y) \) and \( N(x, y) \) are \( C^1 \) (that is continuous with continuous first partial derivatives) inside and on \( C \) then
\[ \oint_{C=\partial A} M \, dx + N \, dy = \iint_A \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy \]

It follows that
\[ \oint_C z \, dz = \oint_C (x - iy)(dx + idy) = \oint_C x \, dx + y \, dy + i \oint_C x \, dy - y \, dx = \iint_A \left( \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) \, dx \, dy + i \iint_A \left( \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} \right) \, dx \, dy = 2i \iint_A \, dx \, dy = 2iA \]

(b) The area \( A \) is bounded by the closed contour \( C = \gamma_1 + \gamma_2 \) where the smooth curves \( \gamma_1 \) and \( \gamma_2 \) are parametrized by
\[ \gamma_1: \quad z(x) = x + iy = x + i(1 - x^2), \quad z'(x) = 1 - 2ix, \quad x \in [-1, 1] \]
\[ \gamma_2: \quad z(x) = x, \quad z'(x) = 1, \quad x \in [-1, 1] \]

Hence integrating around the contour in the anti-clockwise sense gives
\[ A = \frac{1}{2i} \oint_C z \, dz = \frac{1}{2i} \oint_{\gamma_1} + \oint_{\gamma_2} z \, dz = \frac{1}{2i} \int_{-1}^{1} \left[ -(x - i(1 - x^2))(1 - 2ix) + x \right] \, dx = \frac{1}{2i} \int_{-1}^{1} \left[ 2x - 2x^3 + i(1 + x^2) \right] \, dx = \frac{1}{2i} \left[ x^2 - \frac{x^4}{4} + i \left( x + \frac{x^3}{3} \right) \right]_{-1}^{1} = \frac{1}{2} \frac{8}{3} = \frac{4}{3} \]
(c) Cauchy’s theorem states that if $f(z)$ is analytic in a simply-connected open domain $D$ then for any closed contour in $D$

$$\oint_C f(z)\,dz = 0$$

This does not apply here because the function $f(z) = \overline{z}$ is nowhere analytic.
4. (a) Stating clearly any tests that you use, determine the convergence or divergence of the following series throughout the complex plane and find the radius of convergence $R$ of each series:

(i) \[ \sum_{n=1}^{\infty} (-1)^n 2^{n+1} n z^{2n} \]

(ii) \[ \sum_{n=1}^{\infty} \frac{(n+1)! z^n}{n^n} \]

You can assume Stirling’s formula in the form $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$.

(b) A complex function $f(z)$ is defined by the series

\[ f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)} \]

Show that this function is continuous on $|z| \leq 1$ stating clearly any theorem that you use.

4. Solution.

(a) (i) Applying the ratio test gives

\[ \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+1} 2^{n+2} (n+1) z^{2n+2}}{(-1)^n 2^{n+1} n z^{2n}} \right| = \left| \frac{2(n+1) z^2}{n} \right| \to 2|z|^2 = \rho \]

so the series converges absolutely for $|z| < 1/\sqrt{2}$ and diverges for $|z| > 1/\sqrt{2}$ where $R = 1/\sqrt{2}$ is the radius of convergence. However, on the circle of convergence $|z| = 1/\sqrt{2}$, we have

\[ |a_n| = 2n \not\to 0 \quad \text{as} \quad n \to \infty \]

so the series diverges there by the divergence test.

(ii) From Stirling’s formula $(n+1)! = (n+1)n! \sim \sqrt{2\pi n}(n+1)(n/e)^n$. So applying the root test gives

\[ |a_n|^{1/n} = (2\pi n)^{1/n}(n+1)^{1/n} \frac{n |z|}{e} - \frac{|z|}{e} = \rho \]

so the series converges absolutely for $|z| < e$ and diverges for $|z| > e$ where $R = e$ is the radius of convergence. However, on the circle of convergence $|z| = e$, we have

\[ |a_n| \sim \sqrt{2\pi n} (n+1) \left(\frac{n}{e}\right) \left(\frac{e}{n}\right)^n \sim \sqrt{2\pi n} (n+1) \not\to 0 \quad \text{as} \quad n \to \infty \]

so the series diverges there by the divergence test.

(b) By applying the Weierstrass $M$-test

\[ |a_n(z)| = \frac{|z|^n}{n(n+1)} \leq \frac{1}{n^2} = M_n \]
where $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent harmonic series we see that the original series is absolutely and uniformly convergent on the closed domain $|z| \leq 1$. Since $a_n(z) = \frac{z^n}{n(n+1)}$ is a continuous function for each $n \geq 1$, it follows that the series converges to a continuous function $S(z) = \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}$ on $|z| \leq 1$. 
5. Find the Laurent series about the indicated singularity for each of the following complex functions. Name the singularity in each case and state the region of convergence of each series:

(a) \( f(z) = \frac{z - \sin z}{z^3}; \ z = 0 \) 

(b) \( f(z) = \frac{z}{(z + 1)(z + 2)}; \ z = -2 \) 

(c) \( f(z) = (z - 3) \sin \left( \frac{1}{z + 2} \right); \ z = -2 \)

5. Solution.

(a) Using Taylor series

\[
\frac{z - \sin z}{z^3} = \frac{1}{z^3} \left( z - \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right) \right) = \frac{1}{3!} - \frac{z^2}{3!} + \frac{z^4}{7!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+3)!}
\]

so \( z = 0 \) is a removable singularity and the series converges for all \( z \) by the ratio test.

(b) Let \( w = z + 2 \), then the Laurent series is

\[
\frac{z}{(1 + z)(z + 2)} = \frac{w - 2}{w(w - 1)} = \frac{2 - w}{w} \frac{1}{1 - w} = \frac{2 - w}{w} \left( 1 + w + w^2 + \cdots \right) = \left( \frac{2}{w} + 2 + 2w + 2w^2 + \cdots \right) - \left( 1 + w + w^2 + \cdots \right)
\]

\[
= \frac{2}{w} + 1 + w + w^2 + \cdots = \frac{2}{z + 2} + \sum_{n=0}^{\infty} (z + 2)^n
\]

so \( z = -2 \) is a simple pole and, from the geometric series, the series converges in the annulus \( 0 < |z + 2| < 2 \).

(c) Let \( w = z + 2 \), then the Laurent series is

\[
(z - 3) \sin \left( \frac{1}{z + 2} \right) = \frac{1}{(z - 3)} \sin \frac{1}{w} = (w - 5) \left( \frac{1}{w} - \frac{1}{3!w^3} + \frac{1}{5!w^5} - \cdots \right) - (w - 5) \left( \frac{5}{3!w^3} + \frac{5}{5!w^5} - \cdots \right)
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!((z + 2)^{2n} - 5 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!((z + 2)^{2n+1}}
\]

so \( z = -2 \) is an essential singularity and the series converge for all \( z \neq -2 \) by the ratio test.
6. (a) Show that the function
\[ f(z) = \frac{z}{\cosh\left(\frac{1}{2}\pi z\right)} \]
is meromorphic in the entire complex plane and find its zeros and poles.

(b) Use residue calculus to evaluate the closed contour integral
\[ \oint_{C} \frac{z}{\cosh\left(\frac{1}{2}\pi z\right)} \, dz \]
around the ellipse
\[ C : \quad |z - i| + |z + i| = 4 \]


(a) The function \( f(z) \) is analytic at every point in the complex except at the isolated poles corresponding to the zeros of the denominator at \( z = \pm i, \pm 3i, \ldots \). The function is therefore meromorphic on \( \mathbb{C} \) with a single simple zero at \( z = 0 \).

(b) The integrand \( f(z) \) has simple poles at \( z = \pm i, \pm 3i, \ldots \) but only the poles \( z = \pm i \) lie inside the ellipse. By l’Hôpital, the residues at these poles are

\[
\text{Res}(f, i) = \lim_{z \to i} \frac{z(z - i)}{\cosh\left(\frac{1}{2}\pi z\right)} = \lim_{z \to i} \frac{2z - i}{\frac{1}{2}\pi \sinh\left(\frac{1}{2}\pi z\right)} = \frac{2i}{\pi \sinh\left(\frac{1}{2}\pi i\right)} = \frac{2}{\pi} \\
\text{Res}(f, -i) = \lim_{z \to -i} \frac{z(z + i)}{\cosh\left(\frac{1}{2}\pi z\right)} = \lim_{z \to -i} \frac{2z + i}{\frac{1}{2}\pi \sinh\left(\frac{1}{2}\pi z\right)} = \frac{2i}{\pi \sinh\left(\frac{1}{2}\pi i\right)} = \frac{2}{\pi}
\]

By residue calculus, the integral is thus given by
\[
I = (2\pi i)[\text{Res}(f, i) + \text{Res}(f, -i)] = (2\pi i) \left( \frac{4}{\pi} \right) = 8i
\]