Problem Sheet 3

1. Establish the following differentiation formulae:
   (a) \((\sin z)') = \cos z\)   \hspace{1cm}  (b) \((\cos z)') = -\sin z\)   \hspace{1cm}  (c) \((\sinh z)') = \cosh z\)
   (d) \((\cosh z)') = \sinh z\)   \hspace{1cm}  (e) \((\tan z)') = \sec^2 z\)   \hspace{1cm}  (f) \((\arcsin z)') = (1 - z^2)^{-1/2}\)
   (g) \((\arctan z)') = (1 + z^2)^{-1}\).

2. Use the rules for differentiation to find the derivatives of the following functions:
   (a) \(z^2 + 10z\)   \hspace{1cm}  (b) \(\exp(z^3 - z)\)   \hspace{1cm}  (c) \([\cos(z^2)]^3\)
   (d) \((z^3 + 100)^{-4}\)   \hspace{1cm}  (e) \((\log z)^3\) on the plane minus the negative reals
   (f) \(\sinh(z^e)\).
   (Answers: (a) \(2z + 10\); (b) \((3z^2 - 1)\exp(z^3 - z)\); (c) \(-6z\sin(z^2)(\cos(z^2))^2\); (d) \(-12z^2(z^3 + 100)^{-5}\);
   (e) \(3(\log z)^2/z\); (f) \(e^z\cosh(z^e)\).)

3. For each of the following functions \(f\), find an analytic function \(F\) with \(F' = f\):
   (a) \(f(z) = z - 2\)   \hspace{1cm}  (b) \(f(z) = \frac{z^4 + 1}{z^2}\)
   (c) \(f(z) = \sin z \cos z\)   \hspace{1cm}  (d) \(f(z) = \cosh(2z)\).
   (Answers: (a) \(z^2/2 - 2z\); (b) \(z^3/3 - 1/z\); (c) \((\sin z)^2/2\) or \(-\cos(z^2)/2\); (d) \(\frac{1}{2}\sinh(2z)\)).

4. Show that if \(f\) is analytic on a domain \(D\) and \(g\) is analytic on a domain \(\Omega\) containing the range of \(f\),
   then \(g(f(z))\) is analytic on \(D\), and the chain rule holds: \((g(f(z)))' = g'(f(z))f'(z)\).

5. Let \(f\) be analytic on a domain \(D\) and suppose that \(f'(z) = 0\) for all \(z \in D\). Show that \(f\) is constant on \(D\).

6. Suppose that \(f\) is analytic on a domain \(D\) and \(f'(z) = \alpha f(z)\) for all \(z \in D\), where \(\alpha\) is a constant. Show that \(f(z) = C \exp(\alpha z)\), where \(C\) is a constant. (HINT: consider \(g(z) = e^{-\alpha z}f(z)\) and use the previous exercise for \(g\).)

7. Show that \(h(z) = \bar{z}\) is not analytic on any domain. Show the same for the following functions: \(f(z) = |z|\), \(f(x, y) = 3x^2 + 2x - 3y^2 - 1 + i(6xy + 2y)\), and \(f(x, y) = x \cos x + y \sin y + i(x \sin y - y \cos x)\).

8. Let \(f = u + iv\) be analytic. In each of the following, find \(v\) given \(u\):
   (a) \(u = x^2 - y^2\)   \hspace{1cm}  (b) \(u = \frac{x}{x^2 + y^2}\)
   (c) \(u = 2x^2 + 2x + 1 - 2y^2\)   \hspace{1cm}  (d) \(u = \cosh x \cos y\)
   (e) \(u = \cosh x \cos y\).

9. Let \(\gamma\) be a piecewise smooth simple closed curve, and suppose that \(F\) is analytic on some domain containing \(\gamma\). Show that
   \[
   \int_\gamma F'(z) \, dz = 0.\]

10. In each of the following cases, decide whether or not the given function represents a locally sourceless and/or irrotational flow. For those that do, decide whether the flow is globally sourceless and/or irrotational. Sketch some of the streamlines.
   (a) \(x^3 - 3xy^2 + i(y^3 - 3x^2y)\)   \hspace{1cm}  (b) \(\frac{x}{x^2 + y^2} + i\frac{y}{x^2 + y^2}, \quad x^2 + y^2 > 0\)
   (c) \(\cos x \cosh y + i \sin x \sinh y\)   \hspace{1cm}  (d) \(x^2 - y^2 + 2i xy\)
   (e) \(x^2 - y^2 - 2i xy\)   \hspace{1cm}  (f) \(e^y \cos x + i e^y \sin x\)

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(g) $e^x \cos y + ie^x \sin y$

(h) $\cosh x \cos y + i \sinh x \sin y$.

(Answers: (a) sourceless, irrotational; (b) sourceless, irrotational; (c) sourceless, irrotational; (d) neither; (e) sourceless, irrotational; (f) sourceless, irrotational; (g) neither; (h) neither.)

11. Suppose that $G$ is analytic on a domain $D$ and $f(z) = \overline{G'(z)}$. Show that $f$ represents a sourceless/irrotational flow on $D$. 