Shapley Value

\[ x_i = \sum_{A \subset \text{N}} p_n(A) \left[ v(A \cup \{i\}) - v(A) \right] \]

for \( H(i) := \{ A \subset \text{N} : i \notin A \} \)

and \( p_n(A) := \frac{1! \cdot (n - 1! - 1)!}{n!} \)

What's \( H(i) \)?

E.g. \( N = \{1, 2, 3, 4\} \)

\( H(3) = \) all the subsets of \( N \) that \( i = 3 \) is not in

So, \( H(3) = \emptyset, \{3\}, \{2, 3\}, \{4, 3\}, \{1, 2, 3\}, \{1, 4, 3\}, \{2, 4\}, \{1, 2, 4\} \).

What's \( p_n(A) \)?

E.g. \( n = 4, \quad A = \{1, 3\} \)

So, \( |A| = 2 \), and \( p_n(A) = p_n(A) = \frac{2! \cdot (4 - 2 - 1)!}{4!} \)

\[ = \frac{2! \cdot 1!}{4!} = \frac{1}{12} \]
Suppose three types of planes (Piper Cubs, DC-10s, and 707s) use an airport. A Piper Cub requires a 100-yd runway, a DC-10 requires a 150-yd runway, and a 707 requires a 400-yd runway. Suppose the cost (in dollars) of maintaining a runway for one year is equal to the length of the runway. Since 707s land at the airport, the airport will have a 400-yd runway. For simplicity, suppose that each year only one plane of each type lands at the airport. How much of the $400 annual maintenance cost should be charged to each plane?
Let Player 1 = Piper Club

2 = DC-10

3 = 707.

Define: the value of a coalition is the cost associated with the runway length needed to service the largest plane in the coalition.

\[ v(\emptyset) = 0 \]

\[ v(\{1\}) = 100 \]

\[ v(\{2\}) = v(\{3\}) = 150 \]

\[ v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = v(\{1, 2, 3\}) = 400 \]

\[
\begin{array}{c|ccc}
S & P_3(S) & v(S) & v(S) - v(\emptyset) \\
\hline
\emptyset & 2/6 & 0 & 0 \\
\{1\} & 1/6 & 100 & 100 \\
\{2\} & 1/6 & 150 & 150 \\
\{3\} & 1/6 & 400 & 400 \\
\{1, 2\} & 2/6 & 400 & 400 \\
\{1, 3\} & 2/6 & 400 & 400 \\
\{2, 3\} & 2/6 & 400 & 400 \\
\{1, 2, 3\} & 1/6 & 400 & 400 \\
\end{array}
\]

\[ \chi = \frac{200}{6} \]
Player II's value

<table>
<thead>
<tr>
<th>S</th>
<th>P_3(s)</th>
<th>( V(SU(33)) - V(S) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>{}</td>
<td>( \frac{1}{6} )</td>
<td>150 - 0</td>
</tr>
<tr>
<td>{1}</td>
<td>( \frac{1}{6} )</td>
<td>150 - 100</td>
</tr>
<tr>
<td>{2}</td>
<td>( \frac{1}{6} )</td>
<td>400 - 400</td>
</tr>
<tr>
<td>{1,2}</td>
<td>( \frac{1}{6} )</td>
<td>400 - 400</td>
</tr>
</tbody>
</table>

\[ x_2 = \frac{150}{6} \]

Player III's value

<table>
<thead>
<tr>
<th>S</th>
<th>P_3(s)</th>
<th>( V(SU(33)) - V(S) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>{}</td>
<td>( \frac{1}{6} )</td>
<td>400 - 0</td>
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<tr>
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<td>( \frac{1}{6} )</td>
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<tr>
<td>{2}</td>
<td>( \frac{1}{6} )</td>
<td>400 - 150</td>
</tr>
<tr>
<td>{1,2}</td>
<td>( \frac{1}{6} )</td>
<td>400 - 150</td>
</tr>
</tbody>
</table>

Shapley Value suggests:

Piper Cub pay \$33.33, DC-10 pay \$58.33 and 707 pay \$308.33.
Implication of Solution:

All planes that use a portion of the runway should divide equally the cost of the runway.

So, first 100 yd - shared among 3

\[ \text{\rightarrow each pays } \frac{100}{3} = \$33.33 \]

Next 150 - 200 yd, shared 7: DC-10 and 707

\[ \text{\rightarrow DC-100 pays } \frac{100}{2} + \frac{50}{2} \]
\[ = \$58.33 \]

Next 400 - 150 yd, paid by 707 only,

\[ \text{\rightarrow 707 pays } \$250 + \frac{100}{3} + \frac{50}{2} \]
\[ = \$308.33. \]

Thinking exercise

What if there are 10 Piper Cubs, 5 DC-10s, and 2 707s? What should be the Shapley value for each aircraft?