Multiple comparisons — ANOVA

1. k samples from k populations, \( k \geq 2 \).
   \( H_0: \mu_1 = \mu_2 = \ldots = \mu_k \)
   \( H_1: \) at least 1 pair of means not equal
   Reject \( H_0 \) — which pair(s)?

2. Compare pairs
   \( H_0: \mu_i = \mu_j \) is \( H_1: \mu_i \neq \mu_j \)
   Reject \( H_0 \) if \( |\bar{y}_i - \bar{y}_j| > (t\text{-value}) \times \sqrt{\frac{1}{n_i} + \frac{1}{n_j}} \)
   LSD

\[ \begin{cases} \text{Fisher method: } t\text{-value} = t_{n-k} (1 - \frac{\alpha}{2}) \\ \text{Individual error rate } = \alpha = 0.05 \end{cases} \]

\[ \begin{cases} \text{Tukey method: } t\text{-value} = Q_{k, N-k} (1 - \frac{\alpha}{2}) \\ \text{Family error rate } = \alpha = 0.05 \end{cases} \]

\( k = \text{no. of samples} \)
\( N-k = \text{df error} \)

\( Q_{k, N-k} \): Standardised range distr.

Bonferroni method: \( t\text{-value} = t_{n-k} \left(1 - \frac{\alpha}{2m}\right) \)

\( m \): no. of CIs
Example 7.1.1 (Melon)
The Minitab output is shown on page 83.

(a) Determine which pair of varieties are significantly different, using

i. Fisher method; controls comparison-wise (individual) error rate.

ii. Tukey method; controls experiment-wise (family) error rate.

iii. Bonferroni method;

(b) Construct the simultaneous 95% C.I. for the difference between the mean yield of all varieties, using each of the above three methods.

\[ |\bar{y}_i - \bar{y}_j| > LSD \Rightarrow \text{significant difference} \]

(i) Fisher: \( t_v = 2.086, \ LSD = 5.1811 \)

(ii) Tukey: \( t_v = 2.8001, \ LSD = 6.9548 \)

(iii) Bonferroni: \( t_v = 2.9326, \ LSD = 7.2615 \)

<table>
<thead>
<tr>
<th>C</th>
<th>A</th>
<th>D</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>19.5</td>
<td>20.3</td>
<td>30</td>
<td>37.3</td>
</tr>
</tbody>
</table>
(b) To construct 95% C.I.'s for the differences:

They all have the form:

\[(\bar{y}_i - \bar{y}_j) \pm \text{LSD}\]

where LSD is as calculated in (a).

i. Fisher method:

\[\text{LSD} = 2.086 \times 4.302\sqrt{1/6 + 1/6} = 5.1811\]

\(\text{Different sample sizes} \Rightarrow \text{Different LSD for each comparison}\)

The 95% C.I. for

\[\mu_A - \mu_B:\]
\[(\bar{y}_A - \bar{y}_B) \pm 5.1811\]
\[= (20.33 - 37.33) \pm 5.1811\]
\[= (-22.18, -11.82)\]

\[\mu_A - \mu_C:\]
\[(\bar{y}_A - \bar{y}_C) \pm 5.1811\]
\[= (20.33 - 19.5) \pm 5.1811\]
\[= (-4.35, 6.01)\]

\[\mu_A - \mu_D:\]
\[(\bar{y}_A - \bar{y}_D) \pm 5.1811\]
\[= (20.33 - 30) \pm 5.1811\]
\[= (-14.85, -4.49)\]

Similarly, \(\mu_B - \mu_C: (12.65, 23.01); \quad \mu_B - \mu_D: (2.15, 12.52); \quad \mu_C - \mu_D: (-15.68, -5.32).\]
ii. Tukey method:

\[ \text{LSD} = 2.8001 \times 4.302\sqrt{1/6 + 1/6} = 6.9548 \]

The 95% C.I. for

\[ \mu_A - \mu_B: \quad (\bar{y}_A - \bar{y}_B) \pm 6.9548 \]
\[ = (20.33 - 37.33) \pm 6.9548 \]
\[ = (-23.95, -10.05) \]

\[ \mu_A - \mu_C: \quad (\bar{y}_A - \bar{y}_C) \pm 6.9548 \]
\[ = (20.33 - 19.5) \pm 6.9548 \]
\[ = (-0.12, 7.78) \]

\[ \mu_A - \mu_D: \quad (\bar{y}_A - \bar{y}_D) \pm 6.9548 \]
\[ = (20.33 - 30) \pm 6.9548 \]
\[ = (-16.62, -2.72) \]

Similarly, \( \mu_B - \mu_C: (10.88, 24.79); \quad \mu_B - \mu_D: (0.378, 14.288); \quad \mu_C - \mu_D: (-17.5, -3.5). \)
iii. Bonferroni method:

\[ \text{LSD} = 2.9236 \times 4.302\sqrt{1/6 + 1/6} = 7.2615 \]

(Different sample sizes??)

The 95% C.I. for

\[ \mu_A - \mu_B: \quad (\bar{y}_A - \bar{y}_B) \pm 7.2615 \]
\[ = (20.33 - 37.33) \pm 7.2615 \]
\[ = (-24.26, -9.74) \]

\[ \mu_A - \mu_C: \quad (\bar{y}_A - \bar{y}_C) \pm 7.2615 \]
\[ = (20.33 - 19.5) \pm 7.2615 \]
\[ = (-6.43, 8.09) \]

\[ \mu_A - \mu_D: \quad (\bar{y}_A - \bar{y}_D) \pm 7.2615 \]
\[ = (20.33 - 30) \pm 7.2615 \]
\[ = (-16.93, -2.41) \]

Similarly, \( \mu_B - \mu_C: (10.57, 25.09); \quad \mu_B - \mu_D: (0.07, 14.59); \quad \mu_C - \mu_D: (-17.76, -3.24). \)
Minitab will produce the Fisher and Tukey 95\% C.I.'s if we put in the subcommands. However, the data must be in stacked format and we use the command for stacked data.

\begin{verbatim}
MTB > oneway c5 c6;
SUBC> fisher;
SUBC> tukey.
\end{verbatim}

### One-Way Analysis of Variance

**Analysis of Variance on Yield**

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variety</td>
<td>3</td>
<td>1297.8</td>
<td>432.6</td>
<td>23.37</td>
<td>0.000</td>
</tr>
<tr>
<td>Error</td>
<td>20</td>
<td>370.2</td>
<td>18.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>23</td>
<td>1668.0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Individual 95\% CIs For Mean**

<table>
<thead>
<tr>
<th>Level</th>
<th>N</th>
<th>Mean</th>
<th>StDev</th>
<th>Lower CI</th>
<th>Upper CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>20.333</td>
<td>4.590</td>
<td>11.987</td>
<td>28.680</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>37.333</td>
<td>3.830</td>
<td>30.183</td>
<td>44.483</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>19.500</td>
<td>5.822</td>
<td>10.755</td>
<td>28.245</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>30.000</td>
<td>2.098</td>
<td>21.902</td>
<td>38.098</td>
</tr>
</tbody>
</table>

*Pooled StDev = 4.302*
Tukey's pairwise comparisons

Family error rate = 0.0500 Individual error rate = 0.0111

Critical value = 3.96

Intervals for (column level mean) - (row level mean)

1  2  3

2  -23.955  -10.045
3  -6.122
7.788
24.788

4  -16.622  0.378  -17.455
-2.712  14.288  -3.545

Fisher's pairwise comparisons

Family error rate = 0.192 Individual error rate = 0.0500

Critical value = 2.086

Intervals for (column level mean) - (row level mean)

1  2  3

2  -22.181
-11.819

3  -4.349  12.652
6.015  23.015

4  -14.948  2.152  -15.681
-4.465  12.515  -5.319
Diagram: We can represent the results of multiple comparison by a descriptive diagram.

1. Write down the sample means from the smallest to largest, horizontally.

2. Label the groups above the means.

3. Compare each mean with those to its right, systematically, to see whether the difference is significant. Underline the group of means that are not significantly different.

Example 7.1.2 (Melon)
Draw a diagram to represent the results of the multiple comparison done above.

```
C A D B
19.5 20.3 20.0 37.3

_______  ______
```
7.1.2 Error rates

**Comparisons-wise**

Individual error rate: the confidence we have that each of the CI's, individually, includes the true difference between the relevant μ's.

Since we have a number (k>2) of such CI's, we cannot be so confident that all of them are "correct."

**Experiment-wise**

Family error rate: the confidence we can have that all of the CI's are "correct" simultaneously. This rate is larger than the individual error rate (k>2), sometimes much larger.

Fisher, individual error rate = 0.05

Tukey: family error rate = 0.05

Bonferroni

probability that one or more CI is "incorrect".
7.1. ONE-WAY DATA

7.1.3 Contrasts

- for comparing the average of some population means

Examples:

1. $\mu_1 = \frac{\mu_2 + \mu_3}{2}$?

   Same as $\mu_1 - \frac{\mu_2 + \mu_3}{2} = 0$?

2. $\frac{\mu_2 + \mu_3}{2} > \frac{\mu_3 + \mu_4 + \mu_5}{3}$?

   Same as $\frac{\mu_2 + \mu_3}{2} - \frac{\mu_3 + \mu_4 + \mu_5}{3} > 0$?

Note: compare with the previous simpler case (2 means):

$m_i = m_j$ or $m_i - m_j = 0$.

$L = m_i - m_j$
Definition: Linear contrasts

A contrast \( \psi \) is an expression of the form \( \Sigma_i a_i \mu_i \) where \( \Sigma_i a_i = 0 \).

\[ \psi = a_1 \mu_1 + a_2 \mu_2 + a_3 \mu_3 + \cdots \]

Examples: Are the following contrasts?

\[ \mu_1 - \mu_2, \quad a_1 = 1, \quad a_2 = -1 \quad \Sigma a_i = 0 \quad \checkmark \]

\[ \mu_1 + \mu_2 - \mu_3, \quad a_1 = 1, \quad a_2 = 1, \quad a_3 = -1 \quad \Sigma a_i \neq 0 \quad \times \]

\[ \frac{1}{2} \mu_1 - \frac{1}{2} \mu_2 - \frac{1}{2} \mu_3. \quad a_1 = \frac{1}{2}, \quad a_2 = -1, \quad a_3 = -\frac{1}{2} \]

\[ \Sigma a_i = -1 \quad \times \]

Comparing the average of means is equivalent to testing \( \psi = 0, > 0 \) or \( < 0 \) for a suitable \( \psi \).

\[ H_0: \psi = 0 \]

\[ H_1: \psi > 0 \]

\[ m_1 - \frac{1}{3} m_2 - \frac{1}{3} m_3 - \frac{1}{3} m_4 \]

\[ \frac{m_1 + m_2 + m_3 + m_4}{3} \]
For hypothesis testing on or C.I. for $\psi$:

**estimate of $\psi$ is**

$$\hat{\psi} = \sum_i a_i \hat{\mu}_i = \sum_i a_i \bar{y}_i = \mu_1 \hat{y}_1 + \mu_2 \hat{y}_2 + \cdots + \mu_n \hat{y}_n$$

$$\text{s.e. of estimate is }$$

$$\text{s.e.}(\hat{\psi}) = s \sqrt{\frac{a_1^2}{n_1} + \frac{a_2^2}{n_2} + \cdots + \frac{a_n^2}{n_n}}$$

**Var. of $\hat{\psi}$ is**

$$\text{Var.}(\hat{\psi}) = a_1^2 \text{Var.}(\hat{y}_1) + a_2^2 \text{Var.}(\hat{y}_2) + \cdots + a_n^2 \text{Var.}(\hat{y}_n)$$

$$= \sigma^2 \left( \frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2} + \cdots + \frac{\sigma^2}{n_n} \right)$$

We use the following generic formula for C.I. of $\psi$ and hypothesis-testing on $\psi$:

C.I. is

$$\text{estimate } \pm \text{ (table value)} \times \text{s.e. (estimate)}.$$
Example 7.1.3 (Melon)

Determine whether variety B has higher mean yield than the average of the other 3 varieties, at 0.05 level.

**Question:** \( \mu_B > \frac{\mu_A + \mu_C + \mu_D}{3} \) ?

or \( \mu_B - \frac{\mu_A + \mu_C + \mu_D}{3} > 0 \) ?

\[ H_0 : \psi = 0 \]

\[ H_1 : \psi > 0 \]

**Estimate of \( \psi \):**

\[ \hat{\psi} = \bar{y}_B - \frac{\bar{y}_A + \bar{y}_C + \bar{y}_D}{3} \]

\[ = 37.333 - \frac{1}{3}(20.333 + 19.5 + 30.0) = 14.053 \]

**SE(\( \hat{\psi} \)):**

\[ = 5 \sqrt{\frac{\sigma_A^2}{n_A} + \frac{\sigma_C^2}{n_C} + \frac{\sigma_D^2}{n_D}} \]

\[ = 4.302 \sqrt{\frac{1}{6} + \frac{(-3)^2}{6} + \frac{6^2}{6} + \frac{(-5)^2}{6}} \]

\[ = 2.028 \]

**T-statistic:**

\[ \frac{\text{est.} - H_0\text{ value}}{\text{SE(est.)}} = \frac{14.053 - 0}{2.028} \]

\[ = 6.9295 \]

Null distribution is \( t_{20} \)

**P-value:** \( P(T > 6.9295) < 0.0005 \)

\[ \therefore \text{Reject } H_0 \text{ and conclude..... etc.} \]
7.2. TWO-WAY DATA

7.2 Two-way data

7.2.1 Additive model

- Additive model is appropriate.

- Use multiple comparisons to compare the effects of the different levels to see which pairs are different.

Example (Tomato yield)

<table>
<thead>
<tr>
<th>Factor A: Variety</th>
<th>Factor B Density</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>7.9</td>
</tr>
<tr>
<td></td>
<td>9.2</td>
</tr>
<tr>
<td></td>
<td>10.5</td>
</tr>
<tr>
<td>2</td>
<td>8.1</td>
</tr>
<tr>
<td></td>
<td>8.6</td>
</tr>
<tr>
<td></td>
<td>10.1</td>
</tr>
<tr>
<td>3</td>
<td>15.3</td>
</tr>
<tr>
<td></td>
<td>16.1</td>
</tr>
<tr>
<td></td>
<td>17.5</td>
</tr>
</tbody>
</table>

3x4 factorial experiment $\Rightarrow$ 12 treatment combinations

replication 3
Some summary statistics from Minitab:

MTB > table c2 c3; SUBC> mean c1.

Tabulated Statistics
ROWS: variety    COLUMNS: density

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>ALL</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>16.300</td>
<td>18.100</td>
<td>19.933</td>
<td>18.167</td>
<td>18.125</td>
</tr>
</tbody>
</table>

CELL CONTENTS --
yield: MEAN

Two-way ANOVA output:

MTB > twoway c1 levels in c2 c3;
SUBC> additive.

Two-way Analysis of Variance
Analysis of Variance for yield

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>variety</td>
<td>2</td>
<td>327.60</td>
<td>163.80</td>
<td>166.56</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>density</td>
<td>3</td>
<td>86.69</td>
<td>28.90</td>
<td>18.77</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>Error</td>
<td>30</td>
<td>46.07</td>
<td>1.54</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>35</td>
<td>460.36</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Variety effect is significant. Which pairs are different?
- Planting density effect is also significant. Which pairs are different?
- Do multiple comparisons or construct simultaneous C.I.'s for the difference between the main effects. Which is the best variety? the best planting density?
For constructing a C.I. for $\alpha_i - \alpha_j$:  \[ \text{Row effect difference.} \]

\[
\text{estimate of } \alpha_i - \alpha_j = \hat{\alpha}_i - \hat{\alpha}_j \\
= (\bar{R}_i - \bar{y}) - (\bar{R}_j - \bar{y}) \\
= \bar{R}_i - \bar{R}_j
\]

s.e. of the estimate \[ = s \sqrt{\frac{1}{mc} + \frac{1}{mc}} \]

\[ = s \sqrt{\frac{2}{mc}} \]

because each row has $mc$ observations.

The multiple comparison or multiple C.I. can be done as before with the Fisher method, Tukey method or the Bonferroni method:

\[
\text{LSD} = (\text{table value}) \times s \sqrt{\frac{2}{mc}}.
\]
For constructing a C.I. for $\beta_i - \beta_j$:

estimate of $\beta_i - \beta_j = \hat{\beta}_i - \hat{\beta}_j$

$= (\bar{C}_i - \bar{y}) - (\bar{C}_j - \bar{y})$

$= \bar{C}_i - \bar{C}_j$

s.e. of the estimate $= s \sqrt{\frac{1}{mr} + \frac{1}{mr}}$

$= s \sqrt{\frac{2}{mr}}$

because each column has $mr$ observations.

Here

$$LSD = (\text{table value}) \times s \sqrt{\frac{2}{mr}}.$$
Example 7.2.1 (tomato yield)

(a) Construct 95\% multiple comparison C.I.s to compare the effect of variety, using the Tukey method. Draw a diagram to represent the result.

(b) Carry out multiple comparisons for the effect of planting density, using the Bonferroni method. Draw a diagram to represent the result.

Solution:

(a) Let $\alpha_i$ denote the effect of Variety $i$. For a 95\% C.I. for $\alpha_i - \alpha_j$, we use

$$\text{estimate of } \alpha_i - \alpha_j = \bar{R}_i - \bar{R}_j$$

s.e. of the estimate $= s \sqrt{\frac{2}{mc}} = \sqrt{1.5357 \times 2/12} = 0.5059$.

For Tukey's method,

$$\text{table value} = \frac{Q_{k,df}(1 - \alpha)}{\sqrt{2}}$$

where $k = 3$, the number of varieties, and $df = df$ of $s = 30$. Hence,

$$\text{table value} = \frac{Q_{3,30}(0.95)}{\sqrt{2}} = \frac{3.49}{\sqrt{2}} = 2.4678$$

$$\text{LSD} = 2.4678 \times 0.5059 = 1.2485$$
95% C.I. for

\[ \alpha_1 - \alpha_2: \quad \bar{R}_1 - \bar{R}_2 \pm 1.2485 \]
\[ = (11.333 - 12.208) \pm 1.2485 = (-2.12, 0.37) \]

\[ \alpha_1 - \alpha_3: \quad \bar{R}_1 - \bar{R}_3 \pm 1.2485 \]
\[ = (11.333 - 18.125) \pm 1.2485 = (-8.04, -5.44) \]

\[ \alpha_2 - \alpha_3: \quad \bar{R}_2 - \bar{R}_3 \pm 1.2485 \]
\[ = (12.208 - 18.125) \pm 1.2485 = (-7.166, -4.66) \]

Diagram:

<table>
<thead>
<tr>
<th>Variety</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>11.3</td>
<td>12.2</td>
<td>18.1</td>
</tr>
</tbody>
</table>

Conclusion: No difference between Varieties 1 and 2. Variety 3 has higher mean yield.
(b) Let $\beta_i =$ effect of Planting density $i$. We want to compare $\beta_i$ with $\beta_j$.

\[
\text{estimate of } \beta_i - \beta_j = \bar{C}_i - \bar{C}_j
\]

s.e. of the estimate $= s \sqrt{\frac{2}{mr}} = \sqrt{1.5357 \times \frac{2}{9}} = 0.5842$.

For Bonferroni's method, \quad Tukey's method:

\[
\text{table value} = t_{df}(1 - \alpha/(2m)) \quad k = 4
\]

where $df = df$ of $s = 30$ and

\[
1 - \alpha/(2m) = 1 - 0.05/12 = 0.9958.
\]

Hence,

\[
\frac{3.84}{\sqrt{2}} = \frac{3.84}{1.414} = 2.7153
\]

\[
\text{table value} = t_{30}(0.9958) = 2.8215 \approx 2.7153
\]

\[
\text{LSD} = 2.8215 \times 0.5842 = 1.6488
\]
Comparison:

$\beta_1$ and $\beta_2$: $|\bar{C}_1 - \bar{C}_2| = |11.478 - 14.389| = 2.911 > 1.6483; 1.5863$ - sign. different.

$\beta_1$ and $\beta_3$: $|\bar{C}_1 - \bar{C}_3| = |-4.3| > 1.6483; 1.5863$ - sign. different.

$\beta_1$ and $\beta_4$: $|\bar{C}_1 - \bar{C}_4| = |-2.433| > 1.6483; 1.5863$ - sign. different.

$\beta_2$ and $\beta_3$: $|\bar{C}_2 - \bar{C}_3| = |-1.389| < 1.6483; 1.5863$ - NOT sign. different.

$\beta_2$ and $\beta_4$: $|\bar{C}_2 - \bar{C}_4| = |0.478| < 1.6483; 1.5863$ - NOT sign. different.

$\beta_3$ and $\beta_4$: $|\bar{C}_3 - \bar{C}_4| = |1.867| > 1.6483; 1.5863$ - sign. different.

Diagram:

<table>
<thead>
<tr>
<th>Density</th>
<th>1</th>
<th>4</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>11.48</td>
<td>13.91</td>
<td>14.39</td>
<td>15.78</td>
</tr>
</tbody>
</table>

Conclusion: No difference between Planting densities 2 and 4 or between 2 and 3. All other pairs are significantly different.