1. \( \text{cov}(I_t(f), I_t(g)) = \frac{1}{n} \overline{E I_t(f) I_t(g)} \)

\( \text{since } E I_t(f) = E I_t(g) = 0 \)

assuming for the moment that \( t = T \);

same argument works for \( t < T \)

\[ = E \left[ \sum_{k=1}^{n} X_k (W_{t_k} - W_{t_{k-1}}) \times \sum_{j=1}^{n} Y_j (W_{t_j} - W_{t_{j-1}}) \right] \]

\[ = \sum_{k=1}^{n} E(X_k Y_k (W_{t_k} - W_{t_{k-1}})^2) + \sum_{k \neq j} E \left[ X_k X_j (W_{t_k} - W_{t_{k-1}})(W_{t_j} - W_{t_{j-1}}) \right] \]

using \( \sum_{k=1}^{n} a_k \sum_{j=1}^{n} b_j = \sum_{k=1}^{n} a_k b_k + \sum_{k \neq j} a_k b_j \)

Here

\[ E(X_k Y_k (W_{t_k} - W_{t_{k-1}})^2) \overset{CE4}{=} E(\overline{E} \{-f-\mid F_{t_{k-1}}\}) \]

\[ = E \left[ X_k Y_k E( (W_{t_k} - W_{t_{k-1}})^2 \mid F_{t_{k-1}} \right] \]

\[ \overset{CE3, \text{since } X_k, Y_k \text{ are } F_{t_{k-1}} \text{-measurable}}{=} \left( t_k - t_{k-1} \right) E(X_k Y_k) \]

\[ = E(f_t g_t) \text{ when } t \in [t_{k-1}, t_k). \]
Therefore the first sum equals
\[ \sum_{k=1}^{n} (t_k - t_{k-1}) E(f_{t_{k-1}}, g_{t_k}) = \int_{0}^{t} E(f_s, g_s) \, ds \]

since \( E(f_s, g_s) \) is piece-wise constant on \([t_{k-1}, t_k]\).

As for the second (double) sum, it's zero:

say, for \( j < k \),

\[ E \left( X_k Y_j, (W_{t_k} - W_{t_{k-1}})(W_{t_j} - W_{t_{j-1}}) \right) = E E \left( W_{t_{j-1}} - W_{t_k} \right) \]

\[ = E \left[ X_k Y_j, (W_{t_j} - W_{t_{j-1}}) E \left( W_{t_k} - W_{t_{k-1}} \mid \mathcal{F}_{t_j} \right) \right] \]

\[ \mathcal{F}_{t_{j-1}} \subset \mathcal{F}_{t_j} \subset \mathcal{F}_{t_k} \quad \text{but} \quad \mathcal{F}_{t_j} \subset \mathcal{F}_{t_k} \quad (j < k) \]

\[ = 0 \quad \text{since} \quad E(W_{t_k} - W_{t_{k-1}} \mid \mathcal{F}_{t_k}) = E(W_{t_k} - W_{t_{k-1}}) = 0 \]

by indepence of \( W_{t_k} - W_{t_{k-1}} \) from \( \mathcal{F}_{t_{k-1}} \).

\[ 2. \quad Y_t = f(X_t), \quad \text{where} \quad f(x) = e^x \quad \text{with} \]

\[ f'(x) = f''(x) = e^x = f(x) \quad \text{and} \quad dx = q_t \, dW_t - \frac{1}{2} g_t^2 \, dt \]

\[ q_t = g_t \]
By Ito's formula,
\[ dY_t = f'(X_t) dX_t + \frac{1}{2} f''(X_t) (dX_t)^2 \]
\[ = f(X_t) (q_t dW_t - \frac{1}{2} q_t^2 dt) + \frac{1}{2} f(X_t) q_t^2 dt \]
\[ \text{as } (dX_t)^2 = q_t^2 (dW_t)^2 = g_t^2 dt \]
\[ = f(X_t) q_t dW_t \Rightarrow Y_t = Y_0 + \int_0^t f(X_s) q_s dW_s \]

3.(a) First note that \( S_0 = c = 5 \). Next note that
\[ S_t = f(t, W_t) \text{ with } f(t, x) = 5e^{at + bx} \]
so that \( f_t' = af_t, f_x' = b f_t, f_{xx}'' = b^2 f \).

Hence by Ito's formula,
\[ dS_t = f_t'(t, W_t) dt + f_x'(t, W_t) dW_t + \frac{1}{2} f_{xx}''(t, W_t) (dW_t)^2 \]
\[ = \left( \frac{af(t, W_t) + \frac{b^2}{2} f(t, W_t)}{S_t} \right) dt + \frac{b f(t, W_t)}{S_t} dW_t \]
\[ = (a + \frac{b^2}{2}) S_t dt + b S_t dW_t \]
\[ = 0.2 S_t dt + S_t dW_t. \]

The SDE!
Hence $S_t$ will satisfy the SDE if
\[
\begin{cases}
\alpha + \frac{b^2}{2} = 0.2 \\
b = 1
\end{cases}
\iff \alpha = -0.3, \ b = 1; \ c = 5. \\
\text{I found this before!}
\]

(b) $X_t = \frac{1}{S_t} = 0.2(e^{-0.3t + W_t})^{-1} = 0.2 e^{-0.3t - W_t} \bigg|_{t=0}^{t} = 0.2 x_0$!

Now by Itô's formula, as above,
\[
dX_t = 0.3 X_t dt - X_t dW_t + \frac{1}{2} X_t dt
\]
\[
= 0.8 X_t dt - X_t dW_t, \ \text{bingo.}
\]

4. $X_t = f(W_t)$ with $f(x) = \cos x$; since $f'(x) = -\sin x$, $f''(x) = -\cos x$, by Itô's formula
\[
dX_t = f'(W_t) dW_t + \frac{f''(W_t)}{2} (dW_t)^2
\]
\[
= -\sin (W_t) dW_t - \frac{1}{2} \cos (W_t) dt.
\]