Some solutions to Problem Set 1.

1. (a) \( d \) does not satisfy the first axiom since \( d(-1, 1) = |(-1)^2 - (1)^2| = 0 \).
(b) \( d \) is a metric. (c) \( d \) is a metric. If \( d(x, y) = |\arctan x - \arctan y| = 0 \), then \( \arctan x = \arctan y \), so that \( x = y \) since \( x \mapsto \arctan x \) is a one-to-one function. The remaining axioms are evident.

3. Note that \( d(x, y) \) is half the number of horizontal and vertical steps needed to go from \( x \) to \( y \). The first two axioms are trivially satisfied. To see that \( d \) satisfies the triangle inequality we argue by contradiction and assume that there are \( x, y, z \in \mathbb{R}^2 \) such that
\[
d(x, z) > d(x, y) + d(y, z).
\]
In view of the definition of \( d \), this can happen if:

(1) \( d(x, z) > 0 \) and \( d(x, y) = d(y, z) = 0 \).
(2) \( d(x, z) = 1 \) and \( d(x, y) + d(y, z) = 1/2 \).

In the case (1), \( x = y = z \) since \( d(x, y) = d(y, z) = 0 \). But this contradicts \( d(x, z) > 0 \).
In the case (2), one of the numbers \( d(x, y) \), \( d(y, z) \) is equal to 0 and the other is equal to 1/2. Say \( d(x, y) = 0 \) and \( d(y, z) = 1/2 \). Hence \( x = y \) and so \( d(x, z) = 1/2 \) contradicting \( d(x, z) = 1 \). The “area” of the rectangle I is equal to \( 1/2 \cdot 1/2 = 1/4 \). The “area” of the rectangle II is equal to \( 1 \cdot 1 = 1 \).

4. Clearly, \( d_f(x, y) = 0 \) if and only if \( x = y \) and \( d_f(x, y) = d_f(y, x) \). If \( x, y \) and \( z \in X \), then \( d(x, z) \leq d(x, y) + d(y, z) \) and
\[
d_f(x, z) = f(d(x, z)) \leq f(d(x, y) + d(y, z))
\leq f(d(x, y)) + f(d(y, z)) \quad \text{by (a)}
\leq f(d(x, y)) + f(d(y, z)) \quad \text{by (c)}
= d_f(x, y) + d_f(y, z)
\]
Properties (a)–(c) are clear for \( f(t) = kt \). (Then \( d_f = kd \) is the metric \( d \) rescaled by a factor \( k > 0 \)). The derivatives of \( f(t) = t^\alpha \) and \( f(t) = \frac{t}{1+t} \) are positive for \( t > 0 \) so both functions are increasing. The property (b) is obvious for these two functions. If \( 0 < t \leq s \), then \( (t+s)^{\alpha-1} \leq s^{\alpha-1} \leq t^{\alpha-1} \) (since \( \alpha \leq 1 \)) and
\[
(t+s)^{\alpha} = (t+s)(t+s)^{\alpha-1} \leq (t+s)s^{\alpha-1} \leq s^{\alpha} + t^{\alpha},
\]
hence $f(t + s) \leq f(t) + f(s)$. For $f(t) = \frac{t}{1 + t}$ we have
\[
\frac{t + s}{1 + t + s} = \frac{t}{1 + t} + \frac{s}{1 + t + s} \leq \frac{t}{1 + t} + \frac{s}{1 + s}
\]
so that $f(t + s) \leq f(t) + f(s)$.

6. If $d(x, y) = 0$, then $d_i(x_i, y_i) = 0$, $1 \leq i \leq n$, so that $x_i = y_i$ for $1 \leq i \leq n$. So $x = (x_1, \ldots, x_n) = (y_1, \ldots, y_n)$. Since $d_i(x_i, y_i) = d_i(y_i, x_i)$ for $1 \leq i \leq n$, we have $d(x, y) = d(y, x)$. Let $x, y, z \in \prod_{i=1}^n X_i$. Set $a_i = d_i(x_i, z_i), b_i = d_i(x_i, y_i)$, and $c_i = d_i(y_i, z_i)$. We have $a_i \leq b_i + c_i$ for all $1 \leq i \leq n$, so
\[
\left[ \sum_{i=1}^n a_i^2 \right]^{1/2} \leq \left[ \sum_{i=1}^n (b_i + c_i)^2 \right]^{1/2}.
\]
Using Cauchy’s inequality
\[
\left[ \sum_{i=1}^n (b_i + c_i)^2 \right]^{1/2} \leq \left[ \sum_{i=1}^n b_i^2 \right]^{1/2} + \left[ \sum_{i=1}^n c_i^2 \right]^{1/2},
\]
hence
\[
d(x, z) = \left[ \sum_{i=1}^n a_i^2 \right]^{1/2} \leq \left[ \sum_{i=1}^n b_i^2 \right]^{1/2} + \left[ \sum_{i=1}^n c_i^2 \right]^{1/2} = d(x, y) + d(y, z),
\]
and the triangle inequality follows.

8. First note that for all $x, y \in X$, $\frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} < 1$ for all $n$, so
\[
d(x, y) = \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1,
\]
and $d$ is a well-defined function $d : X \times X \to \mathbb{R}$.

If $d(x, y) = 0$, then $\frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} = 0$ for all $n$. Hence $d_n(x_n, y_n) = 0$ for all $n$ implying $x_n = y_n$ for all $n$. So $x = y$. Clearly, $d(x, y) = d(y, x)$ since $d_n(x_n, y_n) = d_n(y_n, x_n)$. In view of the solution of (4), $\frac{d_n(a, b)}{1 + d_n(a, b)}$, $a, b \in X$, is a metric on $X_n$ and so
\[
d(x, z) = \sum_{n=1}^{\infty} \frac{d_n(x_n, z_n)}{2^n} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \left[ \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} + \frac{d_n(y_n, z_n)}{1 + d_n(y_n, z_n)} \right]
\]
\[
= \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{2^n} + \frac{d_n(y_n, z_n)}{2^n} = d(x, y) + d(y, z).
\]

10. (a) For integers $n, m \geq 1$ we have $d(2n, 2m) = |\frac{1}{2n} - \frac{1}{2m}| = \frac{1}{2} \cdot |\frac{1}{n} - \frac{1}{m}| \leq \frac{1}{2}$. So diam $(P) \leq 1/2$. To see that diam $(P) = 1/2$ note that $d(2n, 2) = \frac{1}{2}(1 - \frac{1}{n}) \to \frac{1}{2}$ as $n \to \infty$. For integers $n, m \geq 0$ we have $d(2n+1, 2m+1) = |\frac{1}{2n+1} - \frac{1}{2m+1}| < 1$. So diam $(\mathbb{N} \setminus P) \leq 1$.

Since $d(2n+1, 1) = 1 - \frac{1}{2n+1} \to 1$ as $n \to \infty$, we have diam $(\mathbb{N} \setminus P) = 1$.

(b) $B(2n, \frac{1}{2n}) = \{ m \in \mathbb{N} \mid |\frac{1}{2n} - \frac{1}{m}| < \frac{1}{2n} \} = \{ m \in \mathbb{N}, n < m \}$.

$B(n, \frac{1}{2n}) = \{ m \in \mathbb{N} \mid |\frac{1}{n} - \frac{1}{m}| < \frac{1}{2n} \} = \{ m \in \mathbb{N} \mid \frac{2m}{n} < m < 2n \}$. 