Some solutions to Problem Set 3.

2. If \( A \) is not bounded i.e., if diam \( A = \infty \), then diam \( \overline{A} = \infty \) since \( A \subset \overline{A} \). Now assume that \( a := \text{diam } A < \infty \). If \( x, y \) are adherent points of \( A \), then there are sequences \( x_n, y_n \in A \) such that \( d(x_n, x) \to 0 \) and \( d(y_n, y) \to 0 \). Then

\[
d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y) \leq a + d(x, x_n) + d(y_n, y).
\]

Taking the limit as \( n \to \infty \) gives \( d(x, y) \leq a \) for all \( x, y \in \overline{A} \), hence diam \( \overline{A} \leq a \). Since it is also true that \( a \leq \text{diam } \overline{A} \) we conclude that \( a = \text{diam } \overline{A} \). In general, diam \( A \neq \text{diam } A^0 \). For example, let \( A = (0, 1) \cap \mathbb{Q} \) with the usual metric. Then diam \( A = 1 \), but \( A^0 = \emptyset \).

3. We claim that if \( (x, y) \in X \times Y \), then it is an adherent point of \( A \times B \). Since \( \overline{A} = X \) and \( \overline{B} = Y \), the point \( x \) is an adherent point of \( A \) and \( y \) is an adherent point of \( B \). Hence there exist sequences \( \{x_n\} \) and \( \{y_n\} \) in \( A \) and \( B \) such that \( x_n \to x \) in \( X \) and \( y_n \to y \) in \( Y \). Consequently, \( (x_n, y_n) \to (x, y) \) in \( X \times Y \) and \( (x, y) \) is an adherent point of \( A \times B \) as claimed.

4. (a) The point \( f(t) \) has coordinates \( \left( \frac{t}{t^2+1}, \frac{t^2}{t^2+1} \right) \) and so \( f(t) = \left( \frac{t}{t^2+1}, \frac{t^2}{t^2+1} \right) \). Since \( g(t) = \frac{t}{t^2+1} \) and \( h(t) = \frac{t^2}{t^2+1} \) are continuous from \( \mathbb{R} \) to \( \mathbb{R}^2 \) with the standard metrics, the function \( f(t) = (g(t), h(t)) \) is continuous. The inverse function \( f^{-1} : X \to \mathbb{R} \) is \( f^{-1}(x, y) = \frac{x^2}{x^2+y^2} \) for \( (x, y) \in X \). Since \( g(x, y) = x \) and \( h(x, y) = 1 - y \) are continuous from \( \mathbb{R}^2 \to \mathbb{R} \) and \( h(x, y) \neq 0 \) for \( (x, y) \in X \), the function \( f^{-1}(x, y) = \frac{g(x,y)}{h(x,y)} \) is continuous from \( X \) to \( \mathbb{R} \).

(b) If \( |s_n - s| \to 0 \), then \( \|f(s_n) - f(s)\| \to 0 \) since \( f \) is continuous, and so \( \rho(s_n, s) \to 0 \).

If \( \rho(s_n, s) \to 0 \), then \( \|f(s_n) - f(s)\| \to 0 \) and since \( f^{-1} \) is continuous from \( X \) to \( \mathbb{R} \) we get, by the definition of continuity, \( |f^{-1}(f(s_n)) - f^{-1}(f(s))| = |s_n - s| \to 0 \). Hence both metrics on \( \mathbb{R} \) are equivalent.

5. (a) Since \( |F(f) - F(g)| = |f(0) - g(0)| \leq d_{\infty}(f, g) \), the function \( F \) is continuous from \( (X, d_{\infty}) \) to \( \mathbb{R} \).

(b) The function \( F \) is not continuous when \( X \) is equipped with the metric \( d_1 \). Indeed, define \( f_n(x) = -nx + 1 \) if \( 0 \leq x \leq 1/n \) and \( f_n(x) = 0 \) for \( 1/n \leq x \leq 1 \). Then \( d_1(f_n, 0) = 1/(2n) \to 0 \) but \( F(f_n) = f_n(0) = 1 \not\to 0 = F(0) \).

6. Assume that \( f \) is continuous. Let \( A \subseteq X \). Then \( \overline{f(A)} \) is a closed subspace of \( Y \), and so \( f^{-1}(\overline{f(A)}) \) is a closed subspace of \( X \). Since \( f^{-1}(\overline{f(A)}) \) is closed and \( A \subseteq f^{-1}(\overline{f(A)}) \), we get \( A \subseteq f^{-1}(\overline{f(A)}) \). Hence \( f(\overline{A}) \subseteq f(f^{-1}(\overline{f(A)})) = \overline{f(A)} \) as required. Conversely, assume that \( f(\overline{A}) \subseteq \overline{f(A)} \) for all \( A \subseteq X \). Let \( B \subseteq Y \) be closed. We have to show that \( A := f^{-1}(B) \) is a closed subset of \( X \). Since \( B \) is closed and \( f(\overline{A}) \subseteq \overline{f(A)} = f(\overline{f^{-1}(B)}) \subseteq \overline{B} = B \), we conclude \( \overline{A} \subseteq f^{-1}(B) \) is closed as required.

(b) Assume that \( f \) is continuous and let \( B \subseteq Y \). Then \( f^{-1}(\overline{B}) \) is closed since \( \overline{B} \) is closed and \( f \) is continuous. Since \( f^{-1}(B) \subseteq f^{-1}(\overline{B}) \), \( f^{-1}(B) \subseteq f^{-1}(\overline{B}) = f^{-1}(\overline{B}) \) and the result follows. Conversely, assume that \( f^{-1}(B) \subseteq f^{-1}(\overline{B}) \) for all \( B \subseteq Y \). Let \( B \subseteq Y \) be closed. Then \( \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) = f^{-1}(B) \) so that \( f^{-1}(B) \) is closed, and \( f \) is continuous.

7. By the triangle inequality,

\[
d(x, a) \leq d(x, y) + d(y, a)
\]

for all \( x, y \in X \). So

\[
d(x, a) - d(y, a) \leq d(x, y).
\]
But $x, y$ are arbitrary, so
\[ d(y, a) - d(x, a) \leq d(y, x) = d(x, y). \]

Combining this with the above inequality gives
\[ -d(x, y) \leq d(x, a) - d(y, a) \leq d(x, y), \] that is, $|d(x, a) - d(y, a)| \leq d(x, y)$.

8.  (a) $f$ is uniformly continuous since $|f(x) - f(y)| \leq |x - y|$ for all $x, y$ by the mean value theorem.
(b) $g$ is not uniformly continuous since if $x_n = 1 - 1/n$ and $y_n = 1 - 1/(2n)$, then $|x_n - y_n| = 1/2n \to 0$ and $|f(x_n) - f(y_n)| = n \to \infty$.
(c) $h$ is uniformly continuous. Let $\varepsilon > 0$. Choose $\delta < \varepsilon^2$. Recall (Problem Sheet 1, Q4) that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for all $a, b \geq 0$. Then if $0 \leq x \leq y$ and $y - x \leq \delta$, then $\sqrt{y} = \sqrt{(y-x) + x} \leq \sqrt{y-x} + \sqrt{x}$, that is, $0 \leq \sqrt{y} - \sqrt{x} \leq \sqrt{\delta} < \varepsilon$.
(d) $k$ is not uniformly continuous since if $x_n = \frac{1}{2\pi n}$ and $y_n = \frac{1}{2\pi n + \pi/2}$ then $|k(x_n) - k(y_n)| = 1$ and $|x_n - y_n| = \frac{\pi}{4n(2\pi n + \pi/2)} \to 0$ as $n \to \infty$.

9.  (a) $f_n$ converges pointwise to 0. Further, $f_n$ attains its maximum at $x_n = \frac{2}{n+2}$. Hence, $0 \leq f_n(x) \leq f_n(x_n) = n \left( \frac{2}{n+2} \right)^2 \left( \frac{n}{n+2} \right)^n \leq \frac{4}{n}$ for $0 \leq x \leq 1$. Since $\frac{4}{n} \to 0$, the sequence $(f_n)$ converges uniformly to the function $f = 0$.
(b) If $x = 0$, then $f_n(0) = 0$. For $0 < x \leq 1$ $f_n(x) = n^2 x(1-x^2)^n \to 0$ as $n \to \infty$, by a standard limit. So the sequence $(f_n)$ converges pointwise to $f = 0$. The convergence is not uniform since $f_n$ attains its maximum at the point $x_n = \frac{1}{\sqrt{2n+1}}$ and $f_n(x_n) \to \infty$ as $n \to \infty$.
(c) If $x = 0$, then $f_n(0) = 0$. For $x > 0$ we have $f_n(x) = n^2 x^3 e^{-nx^2} = \frac{n^2}{(e^2)^n} x^3 \to 0$ as $n \to \infty$, by a standard limit. Hence, the sequence $(f_n)$ converges pointwise to $f = 0$. The function $f_n$ attains its maximum at $x_n = \sqrt{3/(2n)}$. The sequence does not converge uniformly to $f = 0$ since $f_n(x_n) = \frac{n^2}{e^2} x_n^3 e^{-nx_n^2} = (\sqrt{3/2})^3 e^{-3/2} \cdot \sqrt{n} \to \infty$ as $n \to \infty$.

10. Define $g(x) = \lim f(x_n)$ where $(x_n)$ is a sequence of points in $A$ converging to $x$. (Since $(x_n)$ is a Cauchy sequence and $f$ is uniformly continuous, it follows that $f(x_n)$ is a Cauchy sequence in $\mathbb{R}$, hence convergent since $\mathbb{R}$ is complete.) We have to show that $g(x)$ is well-defined, that is, it doesn’t depend on the choice of a sequence $(x_n)$ converging to $x$. If $(y_n)$ is another sequence converging to $x$, then by the triangle inequality $d(x_n, y_n) \to 0$ and this implies that $|f(x_n) - f(y_n)| \to 0$ since $f$ is uniformly continuous. So $\lim f(x_n) = \lim f(y_n)$. Now let $\varepsilon > 0$. Since $f$ is uniformly continuous there is $\delta > 0$ such that $a, b \in A$ and $d(a, b) < \delta$ implies $|f(a) - f(b)| < \varepsilon/2$. Take $x, y \in X$ such that $d(x, y) < \delta/2$. Then there are sequences $(x_n), (y_n) \subset A$ such that $d(x, x_n) \to 0$ and $d(y_n, y) \to 0$. So there is $n \geq N$ such that $d(x, x_n) < \delta/4$ and $d(y_n, y) < \delta/4$. Hence, by the triangle inequality,
\[ d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n) < \delta \]
for all $n \geq N$ so that $|f(x_n) - f(y_n)| < \varepsilon/2$. From this we conclude that
\[ |g(x) - g(y)| \leq |g(x) - f(x_n)| + |f(x_n) - f(y_n)| + |f(y_n) - g(y)| \]
\[ < |g(x) - f(x_n)| + |f(y_n) - g(y)| + \varepsilon/2 \]
and after taking limits as $n \to \infty$ we get
\[ |g(x) - g(y)| \leq \varepsilon/2 < \varepsilon \]
if $d_X(x, y) < \delta/2$. Hence $g$ is uniformly continuous. To see that there is exactly one such function, assume that there are two continuous function $g : X \to \mathbb{R}$ and $h : X \to \mathbb{R}$ such that $f = g = h$ on $A$. Then $g - h = 0$ on $A$ and since $g - h$ is continuous and $A$ is dense, $g - h = 0$ on $X$. So $g = h$. 