The aim of this notes is to give an introduction to Algebraic Topology. We introduce algebraic invariants called singular homology and fundamental groups which help in distinguishing some topological spaces and in studying their structure.

Chapter 1 describes some of the prerequisites from Set Topology and Algebra that we assume.

Chapter introduces identification (or quotient) spaces and identification (or quotient) topology. We describe various familiar spaces that can be built from simple subspaces of Euclidean space. Among these are polyhedra built from simplices. The concept of homotopy and homotopy equivalence are introduced.

Chapter 2 develops singular homology. The development is easy but abstract. A main tool for calculating singular homology called Mayer-Vietoris sequence is described.

Chapter 3 is the core of the notes and describes many striking applications of homology theory. These include Brouwer’s fixed point theorem, Brouwer’s invariance of dimension theorem, Alexander-Brouwer’s separation theorem and a special case of Alexander duality. We also briefly mention cohomology.
Chapter 4 discusses fundamental groups and covering spaces. Alexander Duality shows that complement of knoth in $\mathbb{R}^3$ all have the same homology groups. Fundamental group provides a finer invariant to distinguish knoths. It is a basic tool in Topology and Group Theory. But fundamental groups are hard to calculate and covering space theory, a useful topic by itself, provides the calculation in some basic cases.

The course notes is modeled after an one month summer course organized by T.I. F. R. in the 1960s. I have expanded on the number of topics to suit a one semester course.

There are a number of excellent books on this material. For collateral reading, I suggest:
- Armstrong’s Basic Topology for chapters 6 and 7,
- Lee’s Topology of manifolds for chapters 1 and 2,
- Lee’s book and Greenberg & Harper for chapters 2 and 3,
- and all the above books for chapter 4.
We assume some knowledge of metric spaces, abstract topology and abelian groups. We recall some of the prerequisites. See M. Armstrong's 'Basic Topology' chapters 1, 2, 3 for the topological prerequisites & Hartley-Hawken for abelian groups.

A set $x$ and $y$ a set of subsets of $x$. $y$ is said to be a topology on $x$ and elements of $y$ called open subsets of $x$. A $y$ is closed under finite intersections and arbitrary union.

If $x = x \in N(x)$ and $y$ open with $x \in U \in N(x)$, then $N(x)$ is called a neighborhood of $x$ in $x$. Complements of open sets in $x$ are called closed.

If $A \subseteq x$, $x \in x$ is called a limit point of $A$ if every neighborhood (wbd. for what) of $y$ intersects $A \setminus \{y\}$.

Exercise: Show that a subset $B$ of $A$ is closed if and only if $B$ contains all its limit points.

The smallest closed set containing $A$ (that is the intersection of all closed sets containing $A$) is called the closure of $A$. Notation: $\bar{A}$

If $A \subseteq x$, then $B = \{A \cup U | U \subseteq y\}$ in a topology on $x$. The subset $A$ with this topology (called induced topology) is often called a subspace of $A$. If we want to use other topology on $A$, we have to specify it.
From metric spaces, we are used to thinking of open sets $A$ as those which contain some ball for every point $a \in A$. In subspace topologies, balls may not look like open balls in $\mathbb{R}^n$.

**Continuity:** Our view of continuity from functions $f: \mathbb{R} \to \mathbb{R}$ in $\mathbb{R}$ if $x_n \to x$, then $f(x_n) \to f(x)$.

If $f: X \to Y$ is a map of topological spaces, we want to extend this abstract concept. We would like to think of $f$ as continuous, i.e., $A \subseteq X$, if $x$ is a limit point of $A$, then $f(x)$ is a limit point of $f(A)$. This is equivalent to saying that $\overline{f(A)} \subseteq f(\overline{A})$. This statement translates to: if $A$ is closed, then $f^{-1}(A)$ is closed. Hence, $f^{-1}$ is open. Thus,

**Def.** A map $f: X \to Y$ of topological spaces is said to be continuous if $f^{-1}(U)$ is open for every open set $U \subseteq Y$.

If $f$ is 1-1, onto and both $f$, $f^{-1}$ are continuous, $f$ (as well as $f^{-1}$) is said to be a **homeomorphism**. One of the goals in topology is clearly a 'reasonable' class of spaces up to homeomorphism. In this course, we will study various algebraic invariants to distinguish spaces. First, we will look at some
A topological space \( X \) is said to be **Hausdorff** if points of \( X \) can be separated by open sets: i.e., \( x, y \in X, x \neq y \), then there exist open sets \( U_x, U_y \) in \( X \) with \( x \in U_x, y \notin U_x \) and \( U_x \cap U_y = \emptyset \).

\( X \) is said to be **compact** if every open cover of \( X \) admits a finite subcover; that is, given any \( \{ U_i \mid i \in I \} \) with \( X \subseteq \bigcup_{i \in I} U_i \), then there exist a finite subset \( F \) of \( I \) such that

\[
X = \bigcup_{i \in F} U_i
\]

**Exercise:** If \( A, B \) are disjoint compact subsets of a Hausdorff space \( X \), show that an open \( U, V \) in \( X \) with \( A \subseteq U, B \subseteq V \) and \( U \cap V = \emptyset \).

**Exercise:** Show that a compact subset of a Hausdorff space is closed.

**Exercise:** Cut down an image of a compact space in a compact.

We also have some famous results:

**Heine-Borel Lemma:** A closed and bounded subset of \( \mathbb{R}^n \) is compact.

**Exercise:** In H-B Lemma, is \( U \) convex true?

We will sometime use Lebesgue's Lemma:

**Lebesgue's Covering Lemma:** Let \( X \) be a compact metric space and \( \{ U_i \mid i \in I \} \) an open cover of \( X \).

Then \( \exists \delta > 0 \) such that any subset of \( X \) of diameter \( \leq \delta \) is contained in some \( U_i \).
We also need the concept of a product space. Suppose \( X, Y \) are topological spaces. We want to say that \( X \times Y \) is a topological space by taking an open\( \cup \) union of the form \( U \times V \) with \( U \) open in \( X \), \( V \) open in \( Y \). The example \( \mathbb{R}^2 = \mathbb{R}^1 \times \mathbb{R}^1 \) shows that this is not quite true. Even though the sum of the form \( U \times V \) are closed w.r.t. finite intersections, they are not closed under unions. So, we just take a topology on \( X \times Y \) by taking our open \( \cup \) union to be arbitrary unions of sets of the form \( U \times V \) with \( U \) open in \( X \) and \( V \) open in \( Y \). The sets of the form \( U \times V \) (with \( U, V \) open in \( X, Y \), respectively) form what is called a basic for a topology on \( X \times Y \) called the **product topology**. An important property which characterizes the product topology is:

Theorem Consider a map \( f: Z \to X \times Y \) where \( X, Y, Z \) are topological spaces and \( X \times Y \) is equipped with product topology. Let \( p_X: X \times Y \to X \) and \( p_Y: X \times Y \to Y \) be projections.

- If \( f \) is continuous, then \( p_X \circ f \) and \( p_Y \circ f \) are continuous.

We have the following theorem (which we will frequently use):

**Theorem** \( X \times Y \) is Hausdorff if \( X, Y \) are both Hausdorff
A more difficult result is:

**Theorem:** $X \times Y$ (with product topology) is compact if $X$ and $Y$ are.

We will also need the notions of connectedness and path connectedness.

**Path connected:** $X$ is said to be path connected if for any $x, y \in X$, there is a continuous map $\alpha : [0, 1] \to X$ with $\alpha(0) = x$, $\alpha(1) = y$.

**Connected:** $X$ is said to be connected if $X = A \cup B$, $\emptyset = A \cap B$, with both $A, B$ open in their respective topologies $A = \emptyset$ or $B = \emptyset$.

**Exercise:** Show that if $X$ is path connected, then $X$ is connected.

**Exercise:** Show that a continuous image of a connected space is connected.

**Exercise:** Show that $X \times Y$ is connected if $X, Y$ are connected.

There are some of the abstract notions that we use. We will now go on to more connectedness.

**Final Exercise:** Let $f : X \to Y$ be continuous, 1-1 and onto. If $X$ is compact and $Y$ is Hausdorff, show that $f$ is a homeomorphism.
We need the concepts of group, subgroup, normal subgroup, quotient group etc. Further we need free abelian group, (special case of free modules over \( \mathbb{Z} \)) and abelian homomorphism in \( f : g \) abelian groups:

\[ \text{a set, } F(x) \text{ free abelian on } x, y \]

1) \( x \) generates \( F(x) \) and

2) any set map \( e : x \rightarrow A, A \text{ an abelian group} \)

\( \text{extends to a homomorphism } f(x) \rightarrow A \).

Such \( f(x) \) exist and one unique with isomorphism.

**Ex:**

\( F \) is free abelian and \( g : A \rightarrow F \)

is a surjective homomorphism, then \( v \) is a homomorphism \( f : F \rightarrow A \) such that \( g \circ f \).

Show that \( A \cong F \otimes G \), for some \( G \).

**Thm:**

\( f \) is a free abelian group. Then we can write \( A \cong F \oplus 2_{d_1} \oplus \cdots \oplus 2_{d_k} \).

Where \( F \) is \( f.g \) free abelian, \( 2_{d_i} \cong 2/d_i \).

And \( d_i \) are called the formal coefficients of \( A \).

**Thm:**

Sub group of a free abelian group

\( \cong \mathbb{F} \) for free. (For \( f.g \) case, it follows from the above theorem.)
1. Some standard spaces and constructions

We will mainly concentrate on some standard spaces and constructions. These are relatively simple spaces which we can visualize and have a feel for what it means. For example:

The following proofs (up to 1.4.) are only sketches.

1. $D^n = \{ x | x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \| x \| \leq 1 \}$
   
   unit disc

2. $S^{n-1} = \{ x | x \in \mathbb{R}^n, \| x \| = 1 \}$
   
   unit sphere

3. $\text{unit cylinder in } \mathbb{R}^{n+1}$
   
   $= \{ x = (x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1},
   
   \| x_1^2 + \ldots + x_n^2 + x_{n+1}^2 \| = 1,
   
   -1 \leq x_{n+1} \leq 1 \}$

   This can be thought as a product: $S^1 \times [-1,1]$, or

   Identify in $[-1,1] \times [-1,1]$ with $[-1/2,1/2] \times [-1,1]$ with $[-1/2,1/2] \times [-1,1]$ with

   Though it may intuitively clear that all three definitions of the cylinder give the same space, it is not entirely clear why these are the same. Moreover, how does one discuss continuity of maps of such space? In the above example, may be we can take the first definition and ignore the identification.

   But in the Möbius band, the identification makes necessary.
The example 5 and 6 lead to so-called identification (or quotient) spaces and topologies.

Before going on to them, we discuss some maps and homeomorphisms of the more simple spaces.

We feel that

5

unit ball and unit square are homeomorphic and similarly

6

\( S^1 \) and the boundary of the unit square homeomorphic. Since we are in \( \mathbb{R}^2 \), we expect to use coordinates to find such homeomorphism.

In example 6, try to push from the centre along the ray each point of the circle to the square or conversely. For example: \( \gamma: x \rightarrow \frac{x}{\|x\|} \) is a continuous map of \( \mathbb{R}^2 - \{0\} \to S^1 \). Since \( S^1 \subset \mathbb{R}^2 \) has subspace topology and \( y \) can be considered as a map of \( \mathbb{R}^2 - \{0\} \to \mathbb{R}^2 \) and clearly \( (x_1, x_2) \rightarrow \left( \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \right) \) is continuous since both \( \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \) are continuous on \( \mathbb{R}^2 - \{0\} \).

So, we get a continuous map from \( \partial D \to S^1 \).
We call the restriction $r_0$ of $r_0: D \to S^1$ to $\text{continuum, since it is the restriction of a continuous map}$.

Now $r_0$ is clearly one-to-one. (??)

So, there is a good chance $r_0$ is a homeomorphism.

We can prove this by appealing to the very useful theorems recalled in the preliminaries:

1.1. Theorem: If $X$ is compact and $Y$ Hausdorff, then any continuous 1-1 map of $X$ onto $Y$ is a homeomorphism.

This completes a sketch of the fact that $D$ and $S^1$ are homeomorphic in terms of results and concepts you know.

We need, indeed, topology, product spaces, and projection, compactness ($= closed and bounded in R^m$ and in complete metric spaces).

We can try to use this method that $D$ and $\square$ are homeomorphic. For example:

Any point $D$ is of the form $x, z \in S^1$, and $x$ is a point $\square$.

We can just map $x$ to $x \cdot r_0(x)$

Again the map is defined on the whole of $R^2$ and can be expressed in terms of coordinates.

At this stage, we can see that our arguments are more general:

1. We can do similar things with $R^3$.
2. Why not replace $\square$ by $\diamond$, $\triangle$, $\bigcirc$, $\star$ (at any convex, compact 2-dim object).
We will need a 2-dim object, more in the convex. 2-dim can be taken to mean that it has interior, so we can find small balls within.

Of course, $B$ and $\mathcal{B}$ are like $\mathcal{D}$ and $\mathcal{S}$, only the radius and centre are different. We expect to be able to easily transfer the arguments to thin cases.

We expressed $\mathcal{D}$ and $\mathcal{S}$ as sort of cone over their boundaries. Our use of $\mathcal{D}$ makes both look like

\[ \text{cone at} \sigma \]  

This indicates that the arguments extend even further. But even with cones, they may look different physically.

So here may be all specialization of absolute cones.

The examples $\mathcal{B}$, $\mathcal{S}$, and $\mathcal{D}$ above disclose us lead to the so-called quotient spaces or identification spaces. We will describe them next.