1. Let \( x \) be a compact metric space and \( \{ U_i : i \in I \} \) an open cover of \( x \). Show that there exists a real number \( \delta > 0 \) such that any subset \( A \) of \( x \) with diameter less than \( \delta \) is contained in some \( U_i \).

2. Let \( f : x \to y \) be a continuous onto map. If \( x \) is compact and \( y \) is Hausdorff, show that \( f \) is a homeomorphism.

3. Give an example of an identification map \( f : x \to y \) such that \( x \) is Hausdorff and \( y \) is not.

4. Suppose \( f : x \to y \) is an identification map.
   Let \( A \) be a subset of \( x \) and consider \( A \) and \( f(A) \) with the topologies induced from \( x \) and \( y \) respectively. Show \( f|_A : A \to f(A) \) need not be an identification map.

5. Let \( x, x' \) be two topological spaces, \( \beta, \beta' \) partitions of \( x, x' \) and \( \pi : x \to y, \pi' : x' \to y \) corresponding identification maps. Suppose that \( f : x \to x' \) is a continuous map such that whenever \( x, y \) are \( \beta \) an element of \( \beta' \), then \( f(x), f(y) \) are \( \beta' \) an element of \( \beta' \).
5 continued. Show that \( f \) induces a map \( \tilde{f} : \mathbb{R}^n - \{0\} \to \mathbb{R}^n \) (which takes the class of \( x \) to the class of \( f(x) \)) and that \( \tilde{f} \) is continuous.

6. Exercise 5. Show that \( f \) is continuous. An identification map, so is \( \tilde{f} \).

7. There are three descriptions of the (real) \( n \)-dimensional projective space \( \mathbb{P}^n \).

a. On the unit sphere \( S^n \), identify \( x \) and \(-x\),

b. On \( \mathbb{R}^{n+1} - \{0\} \), identify all points which lie on a line through the origin,

c. Identify antipodal points on the boundary of the unit ball \( D^n \).

Show that all three descriptions give rise to the 'same' space. (That is, all the descriptions give rise 'naturally' homeomorphic spaces).

8. (About \( x \cup y \)). Suppose \( A \subseteq Y \) and \( f : A \to X \) is a continuous map. Then \( x \cup y \) is defined to be the identification space obtained from the disjoint union \( X \cup Y \) by identifying \( y \) with \( f(a) \in X \) for all \( a \in A \) with \( f(a) \in X \). Consider \( \tilde{f} : A \times I \to X \times I \) given by \( \tilde{f}(a, t) = (f(a), t) \), \( a \in A \).

Show that \( (X \times I) \cup (Y \times I) \) is homeomorphic to \( (X \cup Y) \times I \).
Problem Sheet 2 (continued)

9. Let \( C \) be a compact convex set in \( \mathbb{R}^n \) and suppose that the interior of \( C \) is non-empty. Show that the frontier of \( C \) is homeomorphic to \( S^{n-1} \).

10. Show that \( S^{n-1} \) is a deformation retract of \( \mathbb{R}^n \).

11. Let \( C \) be a compact convex set in \( \mathbb{R}^n \) and suppose that \( C \) has interior. Show that the frontier of \( C \) is a deformation retract of \( \mathbb{R}^n - \{p\} \) for any interior point \( p \) of \( C \).

(Hint: May assume \( C \) contains unit sphere \( S^{n-1} \) with its interior.)

12. Show that \( \square \) and \( \square \) are deformation retracts of \( \square \).
Problem sheet 1

1. For each \( x \in X \), choose an open ball \( B_x \)
   with center \( x \) that is contained in some \( U_i \).
   Let \( \delta_x \) be the radius of \( B_x \). Choose now
   \( B_x(\delta_x/2) \), a ball with center \( x \) and
   radius \( \delta_x/2 \). Since \( X \) is compact, we
   have a finite number of balls,
   \[ B_{x_1}(\delta_{x_1}/2), \ldots, B_{x_n}(\delta_{x_n}/2) \]
   which cover \( X \).

   Let \( \delta = \frac{1}{2} \min \left( \frac{\delta_{x_1}}{2}, \ldots, \frac{\delta_{x_n}}{2} \right) \)
   and let \( A \) be any subset of \( X \) of diameter \( < \delta \).

   Take some \( a \in A \) and \( a \in B_{x_i}(\delta_{x_i}/2) \) for some \( i \).

   Since \( \text{diam} A < \delta \), \( A \subseteq B_{x_i}(\delta_{x_i}) \). And by
   (since \( \text{diam} A < \delta \), \( A \subseteq B_{x_i}(\delta_{x_i}) \). And by
   a combination \( B_{x_i}(\delta_{x_i}) \subseteq U_j \) for some \( j \).

   So \( A \)

   \text{is contained in some } U_j.

2. \( f: X \to Y \) continuous, 1-1, onto. If \( X \) is compact
and \( Y \) is Hausdorff, then \( f \) is a homeomorphism.

   Let \( A \) be a closed subset of \( X \). \( A \) is compact
   so \( f(A) \) is compact. Since \( X \) is Hausdorff, \( f(A) \)
   is closed. Then image of a closed set is closed.

   Since \( f \) is already 1-1, onto, let see that \( f^{-1} \)
   is also continuous. So \( f \) is a homeomorphism.
Take $X = \mathbb{R}$, and $Y$ the identification space obtained by identifying the rational numbers.

The space $X/\sim$ of corch of $Y$ in $X$ given a similar example.

$f : I \to S^1$, $I = [0, 1]$, $f(x) = e^{2\pi i x}$

is an identification map, since $I$ in compact and $S^1$ is Hausdorff.

Consider $A = (0, 1) \subset I$

$f(A) = S^1$, $f|A \cong [0, 1]$

$f|A$ is not an identification map.

Since $(0, 1)$ is not compact, but $S^1$ is.

$X \xrightarrow{\pi} X'$

$Y \xrightarrow{f} Y'$

Equalities:

$\pi^{-1}(f^{-1}(U)) = (\pi' \circ f')^{-1}(U)$

$= (\pi' \circ f')^{-1}(U)$

$(\pi')^{-1}(U)$ is open since $\pi'$ is continuous, and since $f'$ is continuous, $f'^{-1}(U)$ is open.

Since $\pi$ is identification map, $f^{-1}(U)$ is open.

So $\pi^{-1}(f^{-1}(U))$ is open. So $f$ is continuous.

Proof similar to (5)
Problem 7: Call the three spaces defined in 7 a), b), c) $P^a$, $P^b$, $P^c$. We want to show that all these spaces are homeomorphic.

$P^a$ is obtained by identifying antipodal points of $S^n$: consider $P^c$ obtained from $D^n$ by identifying antipodal points of $S^{n-1} \subseteq D^{n-1}$. We have identification maps and natural maps $i$, $\bar{i}$:

$$
\begin{align*}
D^n \xrightarrow{i} S^n \\
\bar{i} : P_c \rightarrow P^a \\
P^b \rightarrow P^a
\end{align*}
$$

Here $i : D^n \rightarrow S^n$, is the natural map given by $i(x_1, \ldots, x_n) = (x_1, \ldots, x_n, \sqrt{1-\sum x_i^2})$.

$x$ is the inverse of $p(x_1, \ldots, x_n) = (x_1, \ldots, x_n)$.

From this we deduce $\bar{i}$ is continuous and $i$ is continuous. Since $P^b$ is compact and since $\bar{i}$ is continuous, $P^b$ is Hausdorff. $\bar{i}$ is a homeomorphism.

We have maps:

$$
\begin{align*}
P^b \xrightarrow{f} S^n \xrightarrow{i} P^a \\
\bar{i} : P^c \rightarrow P^a
\end{align*}
$$

All maps are continuous and therefore $\bar{i}$ is 1-1 and continuous. $P^b$ is compact, since $S^n \subseteq IR^n - f(X)$ and $\bar{i}(S^n) = \bar{i}(P^b)$, thus $P^b$ is the continuous image of a compact set. So $\bar{i}$ is again a homeomorphism.
We may assume that the origin $O$ is an interior point of $C$. Let $D^r$ be a closed ball of radius $r$ with centre $O$ which is contained in the interior of $C$. Let $S^{n-1}$ be the sphere of a radius $e$ and centre $O$. We will show that $x \mapsto \frac{ex}{\|x\|}$ gives a homeomorphism from $D^r$ to $S^{n-1}$.

To see this we first consider rays through $O$ (this works for any interior point); a ray through $O$ is a line through $O$ and a fixed point in the set of all points of the form $tx$, $t \geq 0$, for a fixed $x$. If $R_x$ is a ray through $O$ and $x$, $R_x \cap C$ in compact, $R_x \cap C$ through $O$ and $x$, and $x \in R_x \cap C$, then any point on the segment $[0, 4]$ must be in $C$, since $C$ is convex. Since $[0, 4]$ must be in $C$, and $R_x \cap C$ must be a closed subset of $[0, 4]$, the interval ray $[0, x]$ clearly $x$ is in $R_x \cap C$. We claim that all other points of $[0, 1]$ are interior points of $C$. 


Thus $C$ is homeomorphic to $G(1, C)$ by a homeomorphism which is identity on $\delta C$ which taken the vector of $G(1, C)$ to any prescribed interior point of $C$.

Corollary if $p$, $q$ are interior points of $C$, then there is a homeomorphism $h: C \rightarrow C$ which is identity on $C$ and takes $p$ to $q$.

We want to show that $S^{n-1}$ is a deformation retract of $R^n \setminus \{0\}$.

Definition. A map $r: X \rightarrow A$ is called a deformation retract of $X$ if there is a continuous map $r: X \rightarrow A$ such that $r(a) = a$, $a \in A$, and $i_0 \circ r = \text{Id}_X$, where $i$ denotes the inclusion of $A$ in $X$.

If there is a map $r: X \rightarrow A$ such that $r(a) = a$, $a \in A$, then $A$ is called a retract of $X$. The map $r$ is called a retraction.

Consider the map $r: R^n \setminus \{0\} \rightarrow S^{n-1}$ given by $r(x) = \frac{x}{\|x\|}$, $\|x\| = 1$.

Clearly $r$ is a retraction:

$$r(x) = x$$ if $\|x\| = 1$.
To see this consider all segments \([z, y]\) for any point \(y\) in any point on \((0, 1)\), these segments fill up a small around \(y\) (radius \(\varepsilon\), length \(|z - y|\), length \(|z - y|\)).

Theorem 2: Let \(\mathbb{S}^1\) be the unit circle in \(\mathbb{R}^2\) centered at \((0,0)\) with radius 1.

Thus \(x\) is the only frontier point on the ray \(Rx\).

So we have that each ray through \(x\) contains one and only one frontier point. \(\mathbb{R}^n\).

Hence the map \(r: \partial C \to \mathbb{S}^1\)

given by \(x \to \frac{x}{|x|}\) is onto, one-to-one, and clearly continuous. Since \(\partial C\) is compact and \(\mathbb{S}^1\) is Hausdorff, we see that \(\partial C\) and \(\mathbb{S}^1\) are homeomorphic.

Additional material:

The above argument works for any interior point of \(C\) containing \(0\). We can use this to show that \(C\) is homeomorphic to \('\text{the}'\) cone over \(\partial C\).
Proof

Consider the geometric cone over \( DC \) with vertex \( e^{nt+1} \) in \( \mathbb{R}^{n+1} \), called \( G_1 DC \).

Map \( G_1 DC \rightarrow C \) by

\[
A e^{nt+1} + (1 - A) C, \quad C \in DC, \quad 0 \leq A \leq 1
\]

This is just the projection of \( \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n \).

This map is 1-1 on each segment \([c, e^{nt+1}]\) and different segments go to different segments. It is the restriction of a continuous map, hence it is continuous. Since \( G_1 DC \) is compact and \( C \) Hausdorff, we have \( G_1 DC \) and \( C \) are homeomorphic.

We can take a point \( p \) above any vertex \( e^{nt+1} \) and map

\[
[e, u] \rightarrow [c, p], \quad c \in DC
\]

by \( A e + (1 - A) C, \quad 0 \leq A \leq 1 \).
The homotopy between \( \text{Id} \) of \( \mathbb{R}^n - \{0\} \) and \( \gamma : \mathbb{R}^n - \{0\} \to S^{n-1} \) is given by

\[
F(x, t) = \begin{cases} 
(1-t)x, & 0 \leq t \leq 1 \\
\frac{x + t \frac{x}{\|x\|}}{\|x + t \frac{x}{\|x\|}\|}, & 0 \leq t \leq 1
\end{cases}
\]

\( F(x, 0) = x \),
\( F(x, 1) = \frac{x}{\|x\|}. \)

We have to check that \( F(x, t) \in \mathbb{R}^n - \{0\} \).

\[
(1-t)x + t \frac{x}{\|x\|} = 0, \quad 0 \leq t \leq 1
\]

\( \Rightarrow \) contradiction.

\( F(x, 1) \) is on the segment joining \( x \) and \( \frac{x}{\|x\|} \).

\( (ii) \)

If \( p \) is an interior point of a compact convex set in \( \mathbb{R}^n \), we want to show that \( \mathbb{R}^n - \{p\} \) is a deformation retract of \( \mathbb{R}^n - \{p\} \).

May assume \( p = \text{origin} \) in \( \mathbb{R}^n \).

We have \( d \) is a homeomorphism, choose \( d \) maps \( d \)

\[
\mathbb{R}^n - \{p\} \to S^{n-1} \to d\mathbb{R}^n - \{p\}
\]

Call the components of \( f \), \( f(c) = c, c \in d \mathbb{R}^n - \{p\} \).
Consider \( F : \mathbb{R}^n \to \mathbb{R}^n \) given by

\[
F(y, t) = tf(y) + (1-t)y, \quad y \in \mathbb{R}^n, 0 \leq t \leq 1
\]

if \( y \in \mathcal{C}, \ f(y) = y, \)

so \( F(y, t) = y, \ t \in \mathbb{R}^n \),

\[
F(y, 0) = y, \ y \in \mathbb{R}^n
\]

\[
F(y, 1) = f(y), \ \text{and} \ f(y) \in \mathcal{C}
\]

\[
F(y, t) = y, \ y \in \mathcal{C}
\]

So \( \mathcal{C} \) is a deformation retract of \( \mathbb{R}^n \) or \( \mathbb{R}^n - \{y\} \) for any interior point \( y \).

**Note:**

If \( y \in \mathcal{C}, \ F(y, t) = tf(y) + (1-t)y \)

\( y \in [y, f(y)] \) and so \( \mathcal{C} \) is a deformation retract of \( \mathcal{C} - \{f(y)\} \) for any interior point. The map is just joint \( p \) to \( y \) and extend to \( f(y) \) where it meets \( \mathcal{C} \).
Problem 11 is used.

Let $C: \{ \rho \} \rightarrow \mathbb{D}$

be the deformation given before.

$\rho \in \mathbb{E}$, taken $E$ to $D$.

And the deformation from $\rho$ to $\mathbb{D}$,

keeps $E$ in $E'$.

Take a bigger rectangle and take $p$ as shown.

If $F$ is the homotopy from $f$ to $g$, then $h$,

$F: S^1 \times I \rightarrow X$,

$F(x,0) = f(x)$, $F(x,1) = g(x)$,

Now form $X \cup (D^2 \times I)$. We have

$X \cup D^2$, $g$ contained in $X \cup (D^2 \times I)$

Use:

are both deformation retractions of $D^2 \times I$.

$D_1 = \partial E$ - minus top lid (containing right disc).

$D_2 = \partial E$ - minus bottom lid (interior of left disc).
Note that $s^1 \times I$ is a subset of both $D_1$ and $D_2$.
and $\forall x \in D^2$, $\exists F$
and $\exists x \in D^2$, $\exists F$

Since the extra point in $D_1$ are identified to points in $X$,

We have retractions $r_1 : E \rightarrow D_1$, $r_2 : E \rightarrow D_2$

and homotopies $F_1 : E \times I \rightarrow E$, $F_2 : E \times I \rightarrow E$

and homomorphism $r_1, r_2 : I \times E$ given by the previous homomorphism.

Problem: We just use them to define a deformation retraction of $X$.

Deformation retraction of $X$ is $D^2$.

as follows: Define $\bar{F}_1 : (X \times E) \times I \rightarrow X \times E$ by

$\bar{F}_1 (x, t) = x$, $x \in X$

$\bar{F}_1 (y, t) = F_1(y, t)$, $y \in E$

$\bar{F}_1 (y, 1) = y$, $y \in E$

If $y \in s^1 \times I$, $\bar{F}_1 (y, t) = y$, and so

The deformation $\bar{F}_1$ is well defined.

Moreover, $\bar{F}_1 (x, 0) = x$, $\bar{F}_1 (y, 0) = y(1)$.

So $\bar{F}_1$ is a homotopy between the identity map of $X$ and a

retraction which takes $X$ to $X \\ D^2$.

Similarly $F_2$, $\bar{F}_2$ are constructed.
Suppose $h: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a homeomorphism. By combining with a translation of $\mathbb{R}^m$, we may assume that $h(0) = 0$ in $\mathbb{R}^m$. Thus, if $\mathbb{R}^n - \{0\}$ and $\mathbb{R}^m - \{0\}$ are the homeomorphic.

We have already seen that $\mathbb{R}^n - \{0\}$ is homotopy equivalent to $S^{n-1}$. Thus, if $\mathbb{R}^n$, $\mathbb{R}^m$ are homomorphic, then $S^{n-1}$ and $S^{m-1}$ are homotopy equivalent.

By (14), $S^1$, so are should be homotopy equivalent. But $S^1$ is connected and $S^0$ is not, $(S^0$ consist of two points).

(16, 17) are straightforward. For (18), observe that the result is true for a 2-simplex and can be reduced to this case.

An argument for (18) is given on the next page.
Problem 18 Let $\sigma$ be the given simplex
and $x \in \sigma$. Consider $d(x, y)$ for $y$ varies over $\sigma$.

Consider $d(z)$, $z \in [z_1, z_2]$. The minimum in $d(x, z)$
attained when $z$ is the foot of the perpendicular
and on both sides of $x$, the distance is monotonic.
So $d(x, z)$, $z \in [z_1, z_2]$ is a function
in $[z_1, z_2]$ with extreme points $d(x, z_1)$ and $d(x, z_2)$
for $d(x, z)$ is attained on $[d(x, z_1), d(x, z_2)]$ or the
interval $[d(x, z_2), d(x, z_1)]$. In any case the
maximum is attained at $z_1$ or $z_2$.

Thus for $(x, y)$, $y \in \sigma$, the
maximum is attained on the frontier.

Now, if the frontier point is not
a vertex, we can take a segment containing
and containing the frontier point in the interior.
So the max $d(x, y)$ is attained when $y$ is
a vertex. Thus, to find diam $\sigma$, we may
assume $y$ is a vertex. The same argument shows
that the max is attained when $y$ is a vertex. Thus
the maximum is just the max $d(x, y)$ on $x, y$
very over vertices of $\sigma$ or the length of
the longest side of $\sigma$. 