1) Let $K$ be a convex subset of $\mathbb{R}^n$ and $A, B \in K$. Show that the affine simplices $\Delta(A, B)$ and $\Delta(B, A)$ are homologous. 
(Hint: Consider $d(d(A, B) + d(A, A))$.)

2) Let $f_1, f_2: C_\ast \rightarrow D_\ast$, $g_1, g_2: D_\ast \rightarrow E_\ast$ be chain maps of chain complexes. If $f_1, f_2$ are homotopic and $g_1, g_2$ are homotopic, show that $g_1 \circ f_1$ and $g_2 \circ f_2$ are homotopic.
(Hint: Show that $g_1 \circ f_1$ and $g_2 \circ f_2$ are homotopic and make a similar reduction.)

3) Show that the notion of chain maps from $C_\ast \rightarrow D_\ast$ of two chain complexes is an equivalence relation.

4) Let $A \subseteq X$ be a retract, and map $i_A : A \rightarrow X$ denote the inclusion. Show that $(i_A)_\ast : \pi_\ast (A) \rightarrow \pi_\ast (X)$ is a direct summand of $\pi_\ast (X)$, for $p \geq 0$.

5) Suppose $A \subseteq X$ is a deformation retract, and $i_A : A \rightarrow X$ denote the inclusion. Show that $(i_A)_\ast : \pi_\ast (A) \rightarrow \pi_\ast (X)$ is an isomorphism for $p > 0$.

6) Show that the augmented chain complex $C_\ast (X)$ of a contractible space is exact.
1) Suppose that we have a commutative diagram of abelian groups and homomorphisms:

\[
\begin{array}{ccccccccc}
A_1 & \xrightarrow{f_0} & A_2 & \xrightarrow{f_1} & A_3 & \xrightarrow{f_2} & A_4 & \xrightarrow{f_3} & A_5 \\
\downarrow{g_1} & & \downarrow{g_1} & & \downarrow{g_2} & & \downarrow{g_2} & & \downarrow{g_3} \\
B_1 & \xrightarrow{h_1} & B_2 & \xrightarrow{h_2} & B_3 & \xrightarrow{h_3} & B_4 & \xrightarrow{h_4} & B_5
\end{array}
\]

Such that

a) the rows are exact,

b) \( f_2, f_4, f_5 \) are isomorphisms,

c) \( f_1 \) is surjective,

d) \( f_3 \) is injective.

Show that \( f_3 \) is an isomorphism.

2) Let

\[
0 \rightarrow C_1 \xrightarrow{g_1} D_1 \xrightarrow{g_3} E_1 \rightarrow 0
\]

\[
0 \rightarrow C_2 \xrightarrow{g_2} D_2 \xrightarrow{g_5} E_2 \rightarrow 0
\]

be a commutative diagram of chain complexes and chain maps with exact rows.

Show that \((f_1)_* \circ d = d \circ (f_3)_*\).

3) Show that the Mayer-Vietoris sequence holds for reduced homology, assuming the usual Mayer-Vietoris sequence.

4) Use the Mayer-Vietoris sequence to verify that \( H^n(S^1) \cong \mathbb{Z}^2 \), for \( n = 0,1 \) and \( n \neq 0,1 \).
Consider \( d \left( \lambda(A, B, A) + \lambda(A, A, A) \right) \)
\[
= \lambda(B, A) - \lambda(A, A) + \lambda(A, B)
+ \lambda(A, A) - \lambda(A, A) + \lambda(A, A)
= \lambda(B, A) + \lambda(A, B)
\]
Thus \( \lambda(A, B) + \lambda(B, A) \) is a boundary.

(1) \( f_1, f_2 : \mathbb{C}^p \to \mathbb{D}^q \) are homotopic.
So, there is a map \( H : \mathbb{C}^p \to \mathbb{D}^{q+1} \) for \( p > 0 \)
Such that \( H \circ d + d \circ H = f_1 - f_2 \).

Let \( g : \mathbb{D}^q \to \mathbb{E}^q \) be any chain map.
and consider \( g \circ H : \mathbb{C}^p \to \mathbb{E}^{q+1} \).

Next suppose \( f : \mathbb{C}^p \to \mathbb{D}^q \) is a chain map
and \( g_1, g_2 : \mathbb{D}^q \to \mathbb{E}^q \) are two homotopic
chain maps. So \( \exists F : \mathbb{D}^q \to \mathbb{E}^{q+1} \) for
with \( F \circ d + d \circ F = g_1 - g_2 \).

So \( (F \circ f) \circ d + d \circ (F \circ f) = g_1 \circ f - g_2 \circ f \)
Thus \( g_1 \circ f, g_2 \circ f \) are homotopic.
By definition there is a continuous function $\gamma : X \rightarrow A$ such that $\gamma \circ i_A = \text{Id}_A$ and $A \circ \gamma = \text{Id}_X$.

So for any $p$,

$\gamma \circ (\mu A) = \text{Id} : \text{hp}(A) \rightarrow \text{hp}(A)$

$\mu A) = \text{Id} : \text{hp}(X) \rightarrow \text{hp}(X)$. 
Now, the result follows from the algebraic lemma:

Suppose \( f : B \to A \) is a homomorphism of abelian groups such that there is a homomorphism \( g : A \to B \) such that \( f \circ g = \text{Id}_A \). Then \( g(A) \) is a direct summand of \( B \).

To see this, let \( K = \ker f \).

We claim \( B = g(A) \oplus K \).

**Proof that \( g(A) + K = B \):**

Take any \( b \in B \), and consider

\[ b - g(f(h)) \]

We can write \( b = g(f(h)) + b - g(f(h)) \).

Since \( g(f(h)) \in g(A) \), we have that \( b - g(f(h)) \in K \).

Applying \( g \) to this element, we get

\[ g(b - g(f(h))) = g(f(h) - g(f(h))) \]

But \( g \circ f = \text{Id}_A \), so the element

\[ f(h) - g(f(h)) = 0 \]

Thus \( g(A) + K = B \).

So \( b \in g(A) \cap K \),

\[ g(h) = 0 \quad \text{and} \quad b = g(a) \quad \text{for some} \ a \in A \]

Thus \( A = f(h) = f(g(a)) = a \). So \( a = 0 \).

Hence \( B = g(A) \oplus K \).
A contractible space is by def. homotopy equivalent to a point space. Thus if \( X \) is contractible, \( H_p(X) = 0, \ p > 0 \) and \( H_0(X) \cong \mathbb{Z} \).

So \( \cdots \xrightarrow{d} C_1(X) \xrightarrow{d} C_0(X) \xrightarrow{d} 0 \) is exact at all \( i, i > 1 \). Since \( H_0(X) \)

\[
\frac{\mathbb{Z}}{B_0(X)} = \frac{C_0(X)}{d(C_1(X))} \cong \mathbb{Z}
\]

\( H_0(X) \) is generated by \([x] \) for any \( x \in X\).

So the augmentation map

\[
\epsilon : C_0 [x] \rightarrow \mathbb{Z}
\]

defined by \( \epsilon(x) = 1 \) for \( x \in X \)

and extending linearly, has kernel \( d(C_1(X)) \).

So \( \cdots \xrightarrow{d} C_1(X) \xrightarrow{d} C_0(X) \xrightarrow{d} \mathbb{Z} \rightarrow 0 \)

is also exact at the 0th stage.

(a) will be done in class

(b) is already done in class

(c) Recall how \( d_* : H_p(E_x) \rightarrow H_{p-1}(C_x) \)

is defined. Consider :
\[ 0 \rightarrow C_p \xrightarrow{d_1} D_p \xrightarrow{\xi} E_p \xrightarrow{d_2} 0 \]
\[ 0 \rightarrow C_{p-1} \xrightarrow{d_3} D_{p-1} \xrightarrow{\xi} E_{p-1} \xrightarrow{d_2} 0 \]

We start with \([ep]\), \(ep\) a cycle in \(E_p\),

- Take \(d_3 dp \in \xi(dp) = ep\). Then \(d_3 dp \in C_{p-1}\).
- Take \(d_2 dp \in C_p\) a cycle and \(d_2 \xi(dp) = 0\).

So \(d_2 dp = \beta_1(C_{p-1}) \in C_{p-1}\). Then \(C_{p-1}\) is a cycle and we define \(d^*_x [ep] = [ep] \in H_{p-1}(C_{p-1})\).

\[
(f_3)_x ([ep]) = \begin{array}{c}
(f_3) \\
\#(ep)
\end{array} \]

By definition of \((f_3)_x\). Now \(d_x\) \(([(f_3)_x(\#(ep))]\))

\[
\begin{array}{c}
0 \\
C_p \\
D_p \\
E_p \\
\rightarrow 0
\end{array}
\]

\[
\begin{array}{c}
0 \\
C_{p-1} \\
D_{p-1} \\
E_{p-1} \\
\rightarrow 0
\end{array}
\]

Call \((f_3)_x(\#(ep)) = ep\).

\[
\begin{array}{c}
dp \\
D_p \\
E_p \\
ep
\end{array}
\]

\[
\begin{array}{c}
edp \\
D_p \\
E_p \\
ep
\end{array}
\]

If we call \(f_2(dp) = d_2 dp\), then \(\beta_2(d_2 dp) = ep\).

Since we assumed that the diagram is commutative,

since our maps are chain maps,

Since our maps are chain maps,

\[
d(d_2 dp) = f_2(dp)
\]

\[
f_2(dp) = \beta_1(C_{p-1}), \text{ then } \beta_1(dp) = f_2(dp) = d_2 dp
\]

\[
\beta_1(dp) = f_2(dp) = \beta_1(C_{p-1}) \text{ by commutativity.}
\]

\[
\beta_1(dp) = f_2(dp) \Rightarrow f_1(dp) = d_2(dp)
\]
Thus
\[
\delta \star [e_p^\star] = [e_p^\star] f_1(\rho \eta_1) = (f_1 \star \delta) \star [e_p^\star]
\]

since \(e_p^\star = f_3 \star (e_p)\),

\[
\delta \star (f_3 \star [e_p^\star]) = [f_1^\star] \delta \star [e_p^\star]
\]

\[
\circ (\delta \circ f_3)^* = (f_1^\star \delta)^*
\]

\[\text{C}\]

\[
\tau \to 0 \to \tau \to \tau \cup \omega \to \omega \to 0 \to 0
\]

\[
H_1(x) \to H_0(U \cup V) \to H_0(U) \cup H_0(V) \to H_0(X) \to 0
\]

\[
\text{By definition, the common homotopy of the augmented sequence of}\ C_0^*(x) \to \cdots \to 0
\]

\[
H_0(x) = \frac{B_0(x)}{B_0(x)} = \frac{C_0(x)}{B_0(x)}
\]

\[
H_0(x) = \frac{x \in \eta}{B_0(x)}
\]

\[
\text{So we have the following maps:}
\]

\[
\tau \to H_0(x)
\]

\[
\tau \to H_0(x)
\]

\[
\tau \to H_0(x)
\]

\[
\text{Now check more generally}
\]

\[
\begin{array}{cccccc}
A_1 & \rightarrow & B_1 & \rightarrow & C_1 & \rightarrow & D_1 & \rightarrow & E_1 \\
J_f & \rightarrow & J_{f_1} & \rightarrow & J_{f_2} & \rightarrow & J_{f_3} & \rightarrow & J_{f_4} \\
J_{f_5} & \rightarrow & B_2 & \rightarrow & C_2 & \rightarrow & D_2 & \rightarrow & E_2 \\
A_1 & \rightarrow & B_2 & \rightarrow & C_2 & \rightarrow & D_2 & \rightarrow & E_2 \\
\end{array}
\]

\[
\text{with column relations extended, then}
\]

\[
\text{to adjoin}
\]

\[
\text{or to adjoin}
\]
(Except for (ii), try the others only if you find them not too hard)

Supplementary problems with Hints and Solutions

(i) Mayer–Vietoris (Extension):

Support $U, V$ are open subsets of $X$ with $X = U \cup V$. Let $U_1, V_1$ be subrings of $U, V$ such that the inclusion maps induce isomorphisms $U_1 \cap V_1 \to U_1 \cap V_1$, $U_1 \to U$, $V_1 \to V$. For example, this holds if $U_1, V_1$ are all deformation retract of $U \cap V$, $U$, $V$ respectively.

Show that there are connecting homomorphisms

$$d^*: H^p(x) \to H^{p-1}(U_1 \cap V_1),$$

such that

$$(a_1)_* \oplus (b_1)_* \equiv (a_2)_* \cdot (b_2)_* \equiv \ldots$$

$$\to H^p(x) \to H^{p-1}(U_1 \cap V_1) \to H^{p-1}(U_1) \oplus H^{p-1}(V_1) \to H^{p-1}(x) \downarrow d^*$$

is exact.

**Hint:** Define $i^*: H^p(x) \to H^p(U_1 \cap V_1)$ by $i^* \cdot d^*

$$\downarrow i^* \downarrow d^* \downarrow H^p(U_1 \cap V_1)

(iii) Show that for any topological space $x$ and the inclusion map $i: x \times I \to x \times I$

$(I = [0, 1], \partial I = \{0, 1\})$, the kernel of

$$i^*: H^p(x \times I) \to H^p(x)$$

is isomorphic to $H^p(x)$. (In general, if $n$ copies of circles are identified at one point, the identification space is called a wedge of $n$ circles.)
Hint \( H_p(x \times D^2) \cong H_p(x) \oplus H_p(x) \)

Take the "diagonal" map:

\[
H_p(x) \rightarrow H_p(x) \oplus H_p(x) \xrightarrow{\sim} H_p(x \times \mathbb{R}^2).
\]

\[
\lambda \rightarrow (\lambda, -\lambda)
\]

Both \((\iota)_*\) and \((\lambda_2)_*\) are isomorphisms.
So kernel \(= H_p(x) \)

(2) Show that \( H_2(S^1 \times S^1) \cong \mathbb{Z}, \quad H_4(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z} \)

Hint: Use Mayer-Vietoris in the form:

\[
\begin{array}{ccc}
V & \cap & U \\
\downarrow & & \downarrow \\
U \cap V & \cup & U \cup V
\end{array}
\]

(3) Consider the wedge of two circles \( S^1 \vee S^1 \subset S^1 \times S^1 \)

\( S^1 \times S^1 \cup \text{fibers} \) and \( p \) as shown

Show that \( S^1 \vee S^1 \) is a deformation retract of \( S^1 \times S^1 \)
Hint

\[
\begin{align*}
\text{Take } b & \quad \text{and } b' \\
\text{if identification maps} & \\
I \times I \rightarrow (T \setminus \{p, p'\}) & \\
\text{induce a homotopy}
\end{align*}
\]

14. (a) Instead of \( T^2 \setminus \{p, p'\} \), take \( T \setminus \{p, p'\} \).

(b) In the homology \( H_1(S^1 \times S^1) \), show that \( S \) represents 0.

15. Calculate \( H_1(C) \).

Hint: use (14)